MAT385 Final (Spring 2008)

Name:

Directions:

a. All problems are equally weighted. You **must** skip one of problems 1-6, but you **may not** skip problems 7-10. Write "SKIP" clearly on the problem you wish to skip.

b. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning).

c. Indicate clearly your answer to each problem (e.g., put a box around it).

d. **Good luck!**

Problem 1. Translate the following wffs into either propositional or predicate logic, and establish their truth value:

a. Thad is a boy. If Ella loves a boy, then he loves Ella. Ella loves Thad. So there is some boy who loves Ella.

b. If it is gorgeous, or if we didn't forget our umbrella, then we will have the picnic. If we have the picnic then we will get salmonella (from the potato salad). It is gorgeous. Therefore, we will get salmonella. [G,U,P,S]

Problem 2. Take any positive integer. If it is even, divide by 2; if it is odd (but not equal to one), subtract 1, then divide by 2. If it is one, then you are finished. By proceeding in this fashion, you will succeed in writing the number as a sum of powers of 2. For example:

$$
\begin{aligned}
183 &= 2 * 91 + 1 \\
&= 2 * (2 * 45 + 1) + 1 \\
&= 2 * (2 * (2 * 22 + 1) + 1) + 1 \\
&= 2 * (2 * (2 * (2 * 11) + 1) + 1) + 1 \\
&= 2 * (2 * (2 * (2 * (2 * 5 + 1)) + 1) + 1) + 1 \\
&= 2 * (2 * (2 * (2 * (2 * (2 * 2 + 1) + 1)) + 1) + 1) + 1
\end{aligned}
$$

a. (2 pts) Write 183 as a sum of distinct powers of 2.

b. (2 pts) Convince me that all positive integers can be written in this way.

c. (6 pts) Demonstrate that if a positive integer can be written as a sum of distinct powers of 2, then the representation is unique. [Hint: you could proceed by contradiction. Write the two different representations as $2^{j_1} + 2^{j_2} + \ldots + 2^{j_p} = 2^{i_1} + 2^{i_2} + \ldots + 2^{i_q}$ – after cancelling like terms – and ....]

Problem 3.

a. (4pts) Draw an expression tree for the calculation of 183 by the product
$2 * (2 * (2 * (2 * (2 * (2 * 2 + 1) + 1)) + 1) + 1) + 1$.

b. Give the following traversals:

- (2pts) post-order =

- (2pts) pre-order =

c. (2pts) What would be the depth of the expression tree for the calculation of $2^n - 1$, $n > 1$, by a similar representation using the process of **Problem 2** (e.g. $2^3 - 1 = 2 * (2 * 1 + 1) + 1)$.

Problem 4. Consider the adjacency matrix, representing an undirected graph:

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{bmatrix} 0 & \infty & \infty & 6 & 11 & 14 \\ \infty & 0 & 3 & \infty & 8 & 9 \\ \infty & 3 & 0 & \infty & 10 & 14 \\ 6 & \infty & \infty & 0 & 15 & 2 \\ 11 & 8 & 10 & 15 & 0 & \infty \\ 14 & 9 & 14 & 2 & \infty & 0 \end{bmatrix}$$

a. (8pts) Use the Bellman-Ford method to find the shortest paths from node c to all other nodes.

Step 1:		a	b	c	d	e	f
	d						
	s						

Step 2:		a	b	c	d	e	f
	d						
	s						

Step 3:		a	b	c	d	e	f
	d						
	s						

Step 4:		a	b	c	d	e	f
	d						
	s						

Step 5:		a	b	c	d	e	f
	d						
	s						

Step 6:		a	b	c	d	e	f
	d						
	s						

b. (2pts) Give the explicit shortest path from c to f, and from c to a.

Problem 5. Consider the graph G given by the adjacency matrix in **Problem 4**.

a. (2 pt) Carefully draw the graph.

b. (2 pt) Is there a Hamiltonian circuit for this graph, or an Euler path? If so, represent them explicitly.

c. (2 pt) Is the graph planar?

d. (4 pts) Is G isomorphic to the undirected graph given by the following adjacency matrix?

$$\begin{bmatrix} 0 & \infty & 11 & 6 & 14 & \infty \\ \infty & 0 & 10 & \infty & 14 & 3 \\ 11 & 10 & 0 & 15 & \infty & 8 \\ 6 & \infty & 15 & 0 & 2 & \infty \\ 14 & 14 & \infty & 2 & 0 & 9 \\ \infty & 3 & 8 & \infty & 9 & 0 \end{bmatrix}$$

Problem 6. Mortgages are generally constructed in the following way: one takes out a sum of money (P_0), and pays it back in equal monthly payments at an interest rate of r (6%, say, written as a decimal – .06) over n years (15, say).

Now, to figure out the monthly payment ρ, we think like this: at the end of month $i+1$ we owe $P_{i+1} = P_i - \rho + AccruedInterest(\mathrm{P}_i)$, where $AccruedInterest(\mathrm{P}_i) = r/12 * P_i$. Hence,

$$P_{i+1} = (1 + r/12)P_i - \rho$$

a. (5 pts) Write a closed form solution for this recurrence relation for P_m.

b. (5 pts) Given that the mortgage is paid off after $n*12$ months, $P_{n*12} = 0$. Use this information to solve for the payment ρ. [Hint: You'll probably need to use one of our most popular identities to simplify the closed-form expression for P_{n*12}.]

Problem 7. Given the following Karnough map derived from a truth function's canonical sum of products:

	x_1x_2	x_1x_2'	$x_1'x_2'$	$x_1'x_2$
x_3x_4	1	0	0	1
x_3x_4'	1	1	1	1
$x_3'x_4'$	0	1	1	0
$x_3'x_4$	0	1	1	0

a. Find the minimal sum-of-products form.

b. Find the minimal equivalent boolean expression using Quine-McCluskey on the complement (the zeros in the table), and verify that you get the same expression as in the part a.

Problem 8. Consider the following regular set S, which is the union of the following two sets:

- S_1: binary strings that are alternating (e.g. no 0 succeeds a 0, and no 1 succeeds a 1).
- S_2: binary strings that contain an odd number of 0s.

a. (4 pts) Give a good variety of example strings for S_1 and for S_2.

b. (6 pts) Find regular expressions for S_1, for S_2, and for S.

Problem 9. In DNA there are 4 bases, adenine (A), guanine (G), cytosine (C) and thymine (T), each attached to a carbon at one side of a sugar. The bases pair up: A with T, C with G (if all is well, that is), so that a legitimate string of DNA would look like ATCGCGATTAGCCG (for example).

Build a finite state machine (represented by a graph) that will read in base letters one at a time and recognize any **correct** sequence of DNA.

Problem 10. Having built a finite machine to do binary addition, we can't resist the urge to construct a binary subtractor. As a hint, you should recall the four states that we used in the binary adder – the binary subtractor uses roughly the same states. Remember that subtraction is not communtative!

I've proposed the names for the four states (which suggest their function) in the following table. Complete the table, and draw the machine (using symmetry as well as you can).

Present State	Next State				Output
	00	10	01	11	
s_{00}					
s_{10}					
s_{01}					
s_{11}					

Draw the corresponding graph of the machine below: