MAT385 Final (Spring 2009): Boolean Algebras, FSM, and old stuff

Name:

Directions: Problems are equally weighted. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

You may skip one of the "Old Stuff" problems (problem 1-7). Write "skip" clearly on the one you don't want me to grade.

Problem 1.

a. Consider the statement "If the processor is fast then the printer is slow."

   i. Negate the statement:

   ii. Write its converse:

   iii. Write the contrapositive:

b. Use propositional logic to prove the following (use letters C, W, R, S):

   The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore the crop is good and there is a lot of sun.

Problem 2. Predicate logic:

a. Write each sentence as a predicate wff (domain is all earthly objects), where B(x) is "x is a ball", R(x) is "x is round", and S(x) is "x is a soccer ball".

   i. Some balls are round but soccer balls are not.

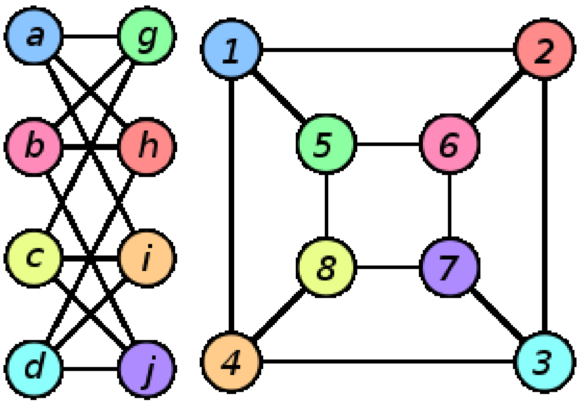
   ii. If soccer balls are round, then all balls are round.

b. Use Predicate logic to prove that

$$(\exists x)P(x) \wedge (\forall x)(P(x) \rightarrow Q(x)) \rightarrow (\exists x)Q(x)$$

Problem 3. Graphs:

a. Determine whether the following graphs are isomorphic or not:



b. Determine whether the graph at left is planar, and, if so, verify Euler's formula for planar graphs.

c. For the graph at right, use depth-first and breadth-first traversals from node 5 to write all the nodes.

Problem 4. Recursion: a ponzi scheme is based on getting new money to pay old customers. Bernie Madoff paid a constant return each year, which tipped off some investigators that he was up to no good. Suppose that Bernie promised to double each person's money each year. He takes a dollar from one person at time $n = 0$. Now he must continue to recruit to keep the scheme going, with customers entering with one dollar. Assume that profits are not reinvested. Suppose he makes no profit himself: that all new money goes to pay old customers.

a. Describe the recruitment schedule as a recursion relation, where by $C(n)$ we mean the number of customers at year n.

b. Write the first five terms of C. Guess the closed form solution.

c. Now check the closed-form solution for $C(n)$.

d. How many years until Bernie Madoff would need more people than there are in the United States today (say 300 million)?

Problem 5. Use the second principle of induction to prove that the n^{th} Fibonacci number $F(n)$ is given by

$$F(n) = \frac{(\varphi_+)^n - (\varphi_-)^n}{(\varphi_+) - (\varphi_-)}$$

where $\varphi_+ = (1 + \sqrt{5})/2$ (the "golden ratio") and $\varphi_- = (1 - \sqrt{5})/2$ are the roots of $x^2 - x - 1 = 0$ (you'll want to use this relationship!).

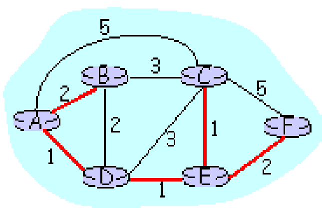
Problem 6. Consider the set of binary trees of depth 2. Assume that single children can be either left or right, and that we distinguish between these two cases.

a. An example is given by the expression tree for the algebraic expression given in prefix notation: + - x 3 * x y. Draw this tree, and give inorder and postorder traversals, hence writing the expression in infix and postfix notation.

b. One equivalence between these trees will be given by their structure: two trees will be considered equivalent if they have the same configuration of arcs. Given this equivalence, how many different trees are there (draw!)?

c. Given the equivalence of part b., consider a second equivalence relation: $T_1 \sim T_2 \iff T_1$ and T_2 have the same number of nodes. What's the minimum number of nodes of a class, maximum number of nodes of a class, and how many distinct trees populate the equivalence class with 5 nodes?

Problem 7. Consider the following graph:



a. Carry out Dijkstra's algorithm to find the shortest route from node A to node F.

b. Find a minimal spanning tree, illustrating your method.

Problem 8. Boolean Algebras

a. State both of DeMorgan's laws using the vocabulary of Boolean Algebras.

b. Prove the following property of Boolean Algebras: $(x \cdot y) + (x' \cdot z) + (x' \cdot y \cdot z') = y + (x' \cdot z)$

Problem 9. Logic Networks

a. Consider the truth function $f(x, y, z) = (x \cdot y) + (x' \cdot z) + (x' \cdot y \cdot z')$ corresponding to the Boolean expression from Problem 8. Write a logic network which evaluates this truth function.

b. Write the corresponding truth table for this function.

c. Use a Karnaugh map to minimize this expression.

Problem 10. Use Quine-McCluskey to find the minimal sum-of-products form for the truth function of Problem 9. [If you haven't gotten the truth table, you can buy it from me for 3 points.]

Problem 11. Finite State Machines

a. Give a regular expression for each of the following sets:

- The set of all strings of 0s and 1s having an odd number of 0s.

- $\{101, 1001, 10001, 100001, \ldots\}$

- The set of all strings of 0s and 1s containing at least one 0.

b. Create a finite state machine to evaluate the truth function of the three preceeding problems (Problems 8-10). Its input is a continuous stream of binary bits, and each time after it receives three bits its output should match the value from the truth table.