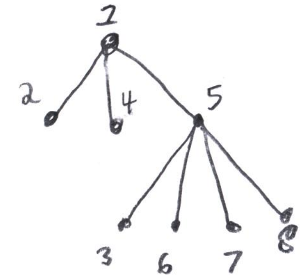
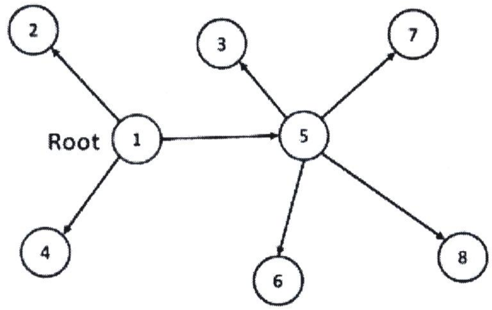
Directions: **You must skip one problem**: write "skip" prominently on it. Show your work! Answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answer to each problem (e.g., put a box around it). **Good luck!**

Problem 1: (10 pts) Consider the following tree:





a. (3 pts) Write the list of nodes resulting from the following traversals (**order children in increasing numerical order**):

- pre-order: 1, 2, 4, 5, 3, 6, 7, 8
- in-order: 2, 1, 4, 3, 5, 6, 7, 8,
- post-order: 2, 4, 3, 6, 7, 8, 5, 1

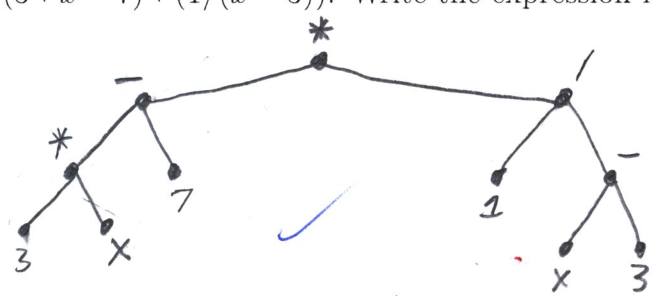
b. (4 pts) At right, draw the expression tree for $(3 * x - 7) * (1/(x - 3))$. Write the expression in both

- prefix notation

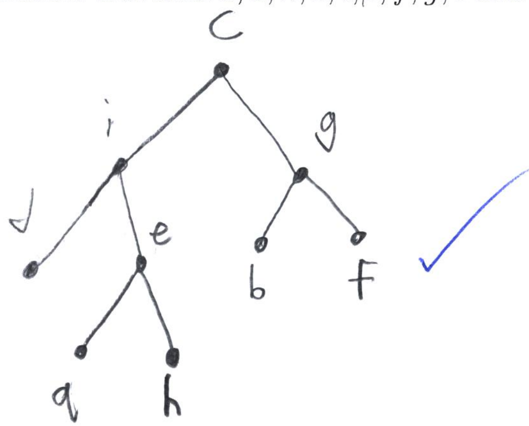
  * - * 3 x 7 / 1 - x 3

- postfix notation

  3 x * 7 - 1 x 3 - / *



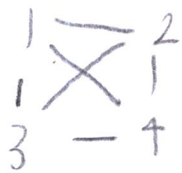
c. (3 pts) Draw a tree that has postorder traversal $d, a, h, e, i, b, f, g, c$ and preorder traversal $c, i, d, e, a, h, g, b, f$.



Problem 2: (10 pts) Consider all possible "Facebooks" that could exist between four distinct individuals.

- (4 pts) How many different friendships are possible? Draw a graph representing the Facebook where all four are friends with the others. What's the name of this graph, and why is its name appropriate?

6 friendships are possible



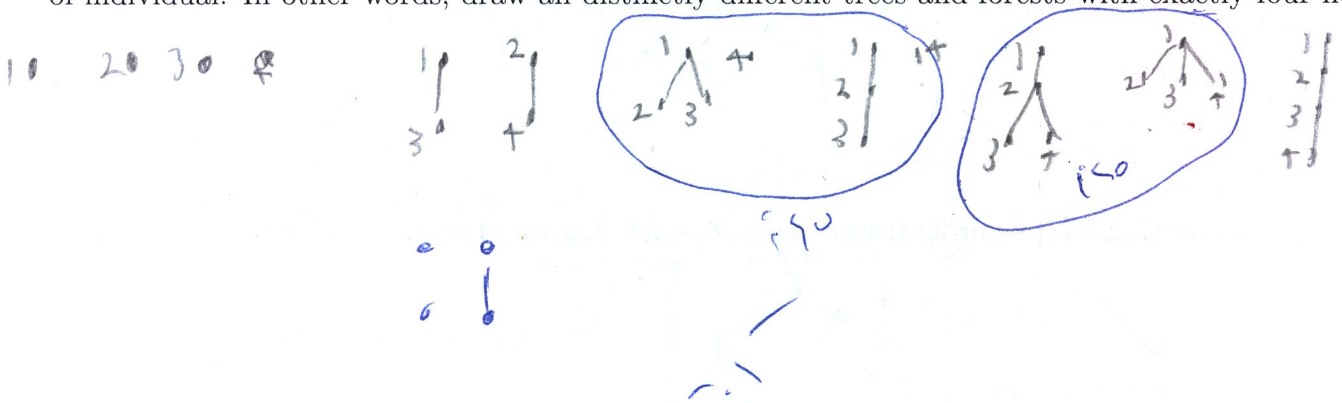
K_4

this is the complete graph, the name is appropriate because all possible arcs (for a simple graph) are included

- (3 pts) How can we use the power set of some set to create all possible Facebooks for these four individuals? How many different Facebooks are possible?

We could use the power set of ordered pairs of the group, There are 6 ordered pairs, so $2^6 = 64$ possible facebooks

- (3 pts) Draw all possible Facebooks that are trees or forests (collections of trees), ignoring distinctions of individual. In other words, draw all distinctly different trees and forests with ~~exactly four nodes~~.



blem 3: (10 pts) We've done some problems with detecting counterfeit coins (one lighter than the hers) with a balance scale. There is a difference in how one approaches the problem depending on whether the number of coins is even or odd.

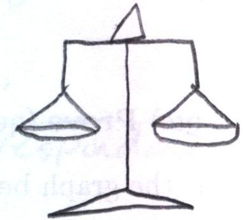
a. Describe your first step if the number of coins n is even.

even = 2, 4, 6, 8, 10 ... 2n

Divide the coins in half and place each half on either side of the scale, then do the same with which ever side is lighter until you find it.

b. Describe your first step if the number of coins n is odd.

odd = 1, 3, 5, 7, 9 ...



Take one coin out of the pile and then put half of the remaining coins on each scale. If it is balanced, the coin you took out is the counterfeit

c. What is the worst case number of weighings if $n = 2^m$, where m is a natural number? Write and solve the recurrence relation (it's easy).

$$W(n) = 1 + W(n/2)$$ ✓

base case

$$W(n) = \log_2(n) = \log_2(2^m) = m$$

$-\frac{1}{2}$

d. What is the depth of the decision tree if the number of coins is $n = 2^m - 1$?

$2^1 - 1 = 1$ no weighing $= 0$

so what the depth?

$2^2 - 1 = 3$ one weighing

$2^3 - 1 = 7$ two weighings

The depth of the decision tree if $n = 2^m - 1$ is $= (m-1)$ ✓

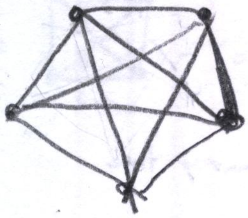
$-\frac{1}{2}$

$m-1$

Problem 4: (10 pts)

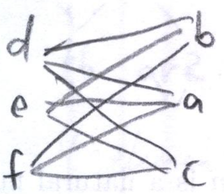
a. (4 pts) All non-planar graphs contain a subgraph isomorphic to one of two graphs: which graphs are they? Draw them and name them.

K_5 $K_{3,3}$

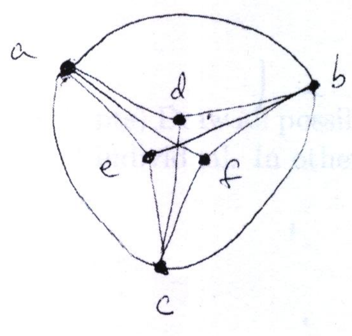


b. (6 pts) **Prove** (quite simply) that

i. the graph below is not planar; and



contains $K_{3,3}$



ii. if we remove any one vertex from the graph, it IS planar.

d, e, f are all degree 3, not 4

There is no complete K_5, and without any 1 vertex, there are not enough vertecies for a $K_{3,3}$

good

Problem 4: (10 pts)

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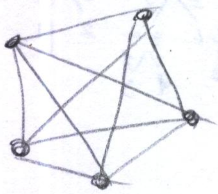
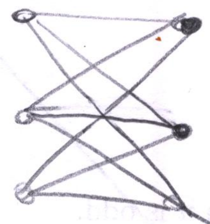
K_5



$K_{3,3}$



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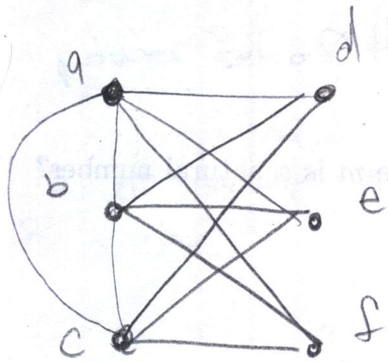
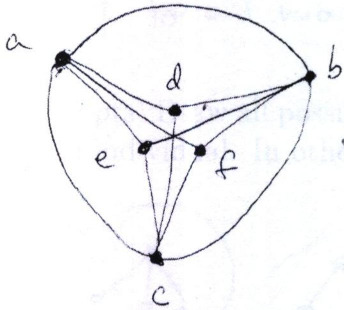


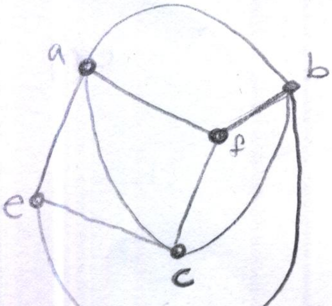


$K_{3,3}$ exists as all arcs except

A ⟷ B
B ⟷ C
C ⟷ A

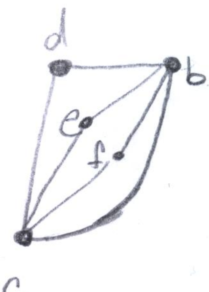
ii. if we remove any one vertex from the graph, it IS planar.

Suppose we remove d (e,f) by symmetry



Nice!

Suppose we remove a (b, c) by symmetry



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oblem 5: (10 pts) Prove that for any finite set S the power set $P(S)$ is larger in size that S itself. (I ight suggest a particular mapping and contradiction; induction would also work.)

Let us assume that for a finite set S P(S) is smaller or the same size as S.

If that is the case, we should be able to construct a one-to-one mapping of elements in P(S) to elements in S (possibly with some elements in S unused).

Nice work

We can see that this fails for small finite sets, e.g.

S	$\{\}$	$\{1\}$	$\{1, 2\}$
P(S)	$\{\{\}, \emptyset\}$	$\{\{1\}, \emptyset\}$	$\{\{1\}, \{2\}, \{1,2\}, \emptyset\}$

But it will also fail for any set of size K by matching elements 1 – K from S to the elements in the power set that are $\{1\}$ – $\{K\}$, the sets of single elements from S. Then, at a minimum, the empty set remains unmatched in P(S). This is a contradiction, which means our original assumption is false. Thus P(S) is larger in ~~size~~ size than S.

It's good you considered the empty set separately.

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Problem 6: (10 pts) I'll give you this much of a reminder of the Euclidean algorithm: $a = qb + r$; then recurse. When $r = 0$, b is the greatest common divisor of the original two numbers.

a. (4 pts) Illustrate the Euclidean algorithm for the computation of the greatest common divisor of the two **successive** Fibonaccis 21 and 34. What is special about the choice of consecutive Fibonaccis with respect to this algorithm?

34 = 1(21) + 13
21 = 1(13) + 8
13 = 1(8) + 5
8 = 1(5) + 3
5 = 1(3) + 2
3 = 1(2) + 1
2 = (1)1 + 1
1 = (1)1 + 0

gcd = 1

consecutive fibonacci #s are the worst case for the algorithm ✓

b. (3 pts) Now consider the Euclidean algorithm for the computation of the greatest common divisor of the two **non**-consecutive Fibonaccis 34 and 13 (with just one "skip" – 21). What do you observe about the very first remainder?

34 = 2(13) + 8
13 = 1(8) + 5
8 = 1(5) + 3
5 = 1(3) + 2
3 = 1(2) + 1
2 = 1(1) + 1
1 = 1(1) + 0

✓ the first remainder is a Fibonacci #
Specifically the one right below the smaller #

c. (3 pts) Prove that this pattern always works with "almost consecutive Fibonaccis", those with only one Fibonacci between them: $F(n)$ and $F(n-2)$.

3 = 3(1) + 0　　5 = 2(2) + 1　　base case
1 = 1(1) + 0　　q = 2

F(n) = qF(n-2) + r

F(n-1) + F(n-2) = 2F(n-2) + r
F(n-1) = (2-1) F(n-2) + r
F(n-1) = F(n-2) + r
F(n-2) + F(n-3) = F(n-2) + r
r = F(n-3) ✓

recursive step
F(n+1) = 2F(n+1-2) + F(n+1-3)
F(n+1) = 2 F(n-1) + F(n-2)
F(n+1) = F(n) + F(n-1)　Fib def
　　　　　　　　　　　Fib def
F(n+1) = F(n+1) ✓

Nice work

Problem 6: (10 pts) I'll give you this much of a reminder of the Euclidean algorithm: $a = qb + r$; then recurse. When $r = 0$, b is the greatest common divisor of the original two numbers.

a. (4 pts) Illustrate the Euclidean algorithm for the computation of the greatest common divisor of the two **successive** Fibonaccis 21 and 34. What is special about the choice of consecutive Fibonaccis with respect to this algorithm? Choice of consecutive Fibonaccis WRT this alg is that is yields the worst case, where r_n decrease slowly. $\frac{r_{n-2}}{r_{n-1}} = 1 \Rightarrow$ relatively prime.

gcd(34,21)=1

Eucl. Alg:

$$\left.\begin{aligned}
34 &= \underline{1} * 21 + \underline{13}\\
21 &= \underline{1} * 13 + \underline{8}\\
13 &= \underline{1} * 8 + \underline{5}\\
8 &= \underline{1} * 5 + \underline{3}\\
5 &= \underline{1} * 3 + \underline{2}\\
3 &= \underline{1} * 2 + \underline{1} \text{ gcd}\\
2 &= \underline{2} * 1 + \underline{0}
\end{aligned}\right\} \text{7 steps}$$

✓

b. (3 pts) Now consider the Euclidean algorithm for the computation of the greatest common divisor of the two **non-**consecutive Fibonaccis 34 and 13 (with just one "skip" – 21). What do you observe about the very first remainder? gcd(34, 13) = 1

Eucl Alg:

$$\left.\begin{aligned}
34 &= \underline{2} * 13 + \underline{8}\\
13 &= \underline{1} * 8 + \underline{5}\\
8 &= \underline{1} * 5 + \underline{3}\\
5 &= \underline{1} * 3 + \underline{2}\\
3 &= \underline{1} * 2 + \underline{1} \text{ gcd}\\
2 &= \underline{2} * 1 + \underline{0}
\end{aligned}\right\} \text{6 steps}$$

the very fist remainder is the 2nd remainder from part a. So, we by pass or "skip" one step. This meanings only slightly better than the worst case scenario

good

c. (3 pts) Prove that this pattern always works with "almost consecutive Fibonaccis", those with only one Fibonacci between them: $F(n)$ and $F(n-2)$.

Now consider

(2) ~~Let~~ F(n) + F(n-2) be "almost consecutive fibonaccis".

Then gcd(F(n), F(n-2)) can be determined by Euclidean Algorithm; a=qb+r.

Let a = F(n) + b = F(n-2).

Then

$$\left.\begin{aligned}
F(n) &= \underline{2} * F(n-2) + \underline{F(n-3)}\\
F(n-2) &= \underline{1} * F(n-3) + \underline{F(n-4)}\\
F(n-3) &= \underline{1} * F(n-4) + \underline{F(n-5)}\\
&\ \ \vdots
\end{aligned}\right\} \text{k-1 steps}$$

How do you know it works?

✓

(1) Let F(n) + F(n-1) be consecutive Fibbonaccis.

Then By Euclid. Alg, Fn = q * F(n-1) + r

$$\left.\begin{aligned}
F_n &= \underline{1} * F(n-1) + \underline{F(n-2)}\\
F(n-1) &= \underline{1} * F(n-2) + \underline{F(n-3)}\\
F(n-2) &= \underline{1} * F(n-3) + \underline{F(n-4)}\\
F(n-3) &= \underline{1} * F(n-4) + \underline{F(n-5)}\\
&\ \ \vdots
\end{aligned}\right\} \text{k steps}$$

Until reach b=0.

Nice work.

By (1), there k steps needed in order to find the gcd(F(n), F(n-1)) = 1. By (2), there are (k-1) steps needed in order to find the gcd(F(n), F(n-2)) = 1. Since k > k-1, then "almost consecutive Fibonacci's" will always skip that first remainder of consecutive Fibs. Thus, always true. ∎

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