Directions: Problems are worth 10 points each. Show your work: answers without justification will likely result in few points. Your written work also allows me the option of giving you partial credit in the event of an incorrect final answer (but good reasoning). Indicate clearly your answers to each problem (e.g., put a box around them). **Good luck!**

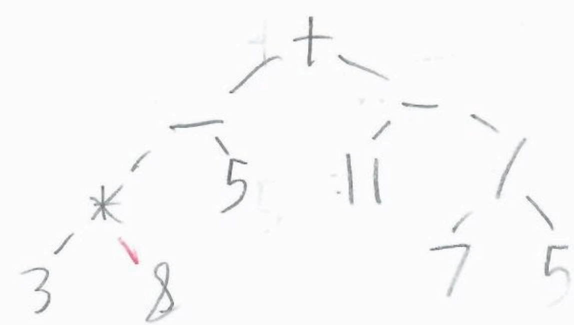
Problem 1:

a. (6 pts) Draw an expression tree whose

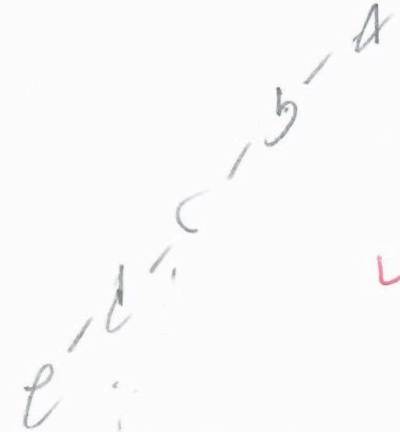
+ - * - /

- preorder traversal is +,-,*,3,8,5,-,11,/,7,5
- postorder traversal is 3,8,*,5,-,11,7,5,/,-,+

3 8 5 11 7 5



b. (2 pts) Draw a binary tree containing the nodes a,b,c,d,e whose inorder and postorder traversals are the same.



I like it.

Better directed?

c. (2 pts) Draw a tree whose in-order, preorder, and postorder traversals are the same.

Problem 1:

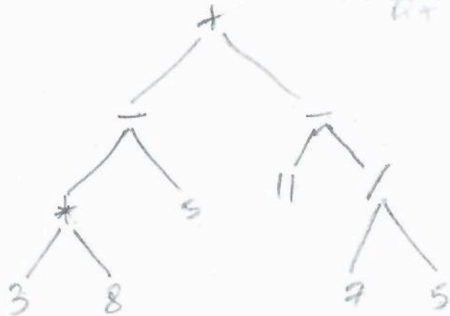
Pre = * 3 8 5
Postorder = 3 8 * 5 -
pre = * 3 8
Post = 3 8 *

Pre =
Post = 5 -

a. (6 pts) Draw an expression tree whose

- preorder traversal is +,-,*,3,8,5,-,11,/,7,5 Root, LR
- postorder traversal is 3,8,*,5,-,11,7,5,/,-,+ L R Root

Pre = - 11 / 7 5
Post = 11 7 5 / -
Pre = / 7 5
Post: 7 5 /



b. (2 pts) Draw a binary tree containing the nodes a,b,c,d,e whose inorder and postorder traversals are the same.

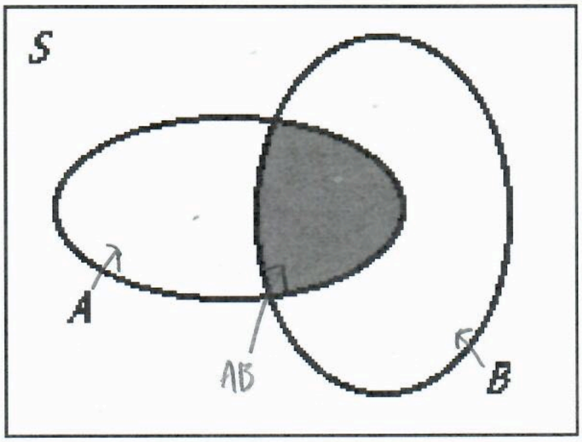
Post: a b c d e L R Root
Inorder: a b c d e L Root R

c. (2 pts) Draw a tree whose in-order, preorder, and postorder traversals are the same.

a

tree with just one node "a"

Problem 2: The Venn diagram of two sets A and B divides the universe S into four distinct regions (actually subsets of S). The dark region below (one of the four regions) is the intersection of A and B:



Good to label, & name things!

a. (4 pts) How many different subsets of S can we form from these four "region sets"? Explain.

16. The powerset size is # of regions squared.

The universe but I will accept you treating it as the fourth region.

$S = \{S, A, B, AB\}$

1 2 3 4 5 6 7 8 9 10 11 12 13 14

$phi(S) = \{\emptyset, \{S\}, \{A\}, \{B\}, \{AB\}, \{S, A\}, \{S, B\}, \{S, AB\}, \{A, B\}, \{A, AB\}, \{B, AB\}, \{S, A, B\}, \{S, A, AB\}, \{S, B, AB\},$

15 16

$\{A, B, AB\}, \{S, A, B, AB\}\}$

b. (6 pts) Some of the sets in part a. contain **exactly two** of the four subsets. (How many are there?) Use the sets A, B, and the binary operations of intersection and union, as well as the unary operation complement, to describe each of these sets. For example, the set shown could be described as $A \cap B$ (although it isn't made up of two of the four subsets).

There are 6: $\{\{S, A\}, \{S, B\}, \{S, AB\}, \{A, B\}, \{A, AB\}, \{B, AB\}\}$

These are described as

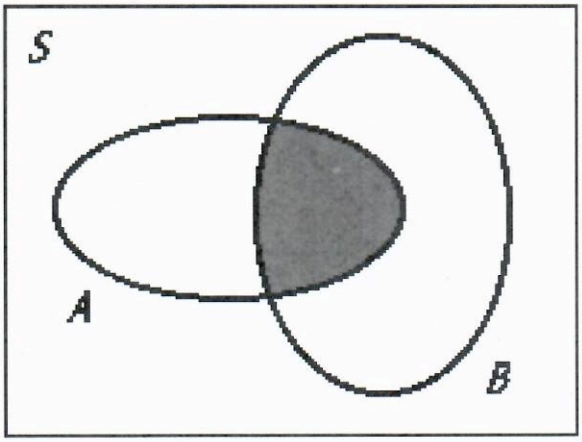
B' A' $(A' \cap B') \cup (A \cap B)$ $((A' \cap B') \cup (A \cap B))'$ A B

$(A' \cap B') = S$

$(A \cap B) = AB$

Very elegant!

Problem 2: The Venn diagram of two sets A and B divides the universe S into four distinct regions (actually subsets of S). The dark region below (one of the four regions) is the intersection of A and B:



A B &

$2^4 = 16$

AB A B X

a. (4 pts) How many different subsets of S can we form from these four "region sets"? Explain.

2^4 or 16 subsets. The four regions are $!A \wedge !B$, $A \wedge !B$, $!A \wedge B$, and $A \wedge B$. Since these regions are exclusive, a subset can be created using any combination of them. For example, a subset may be $(A \wedge !B \,||\, !A \wedge B) \wedge !(A \wedge B) \wedge !(!A \wedge !B)$. Since each of 4 regions either is or isn't included, the # of subsets is 2^4 or 16.

b. (6 pts) Some of the sets in part a. contain **exactly two** of the four subsets. (How many are there?) Use the sets A, B, and the binary operations of intersection and union, as well as the unary operation complement, to describe each of these sets. For example, the set shown could be described as $A \cap B$ (although it isn't made up of two of the four subsets).

The number of sets would be $4C2$, or 6.

1) $(A \cap !B) \cup (A \cap B) = A$

2) $(!A \cap B) \cup (A \cap B) = B$

3) $(!A \cap !B) \cup (A \cap B)$

4) $(A \cap !B) \cup (!A \cap B)$

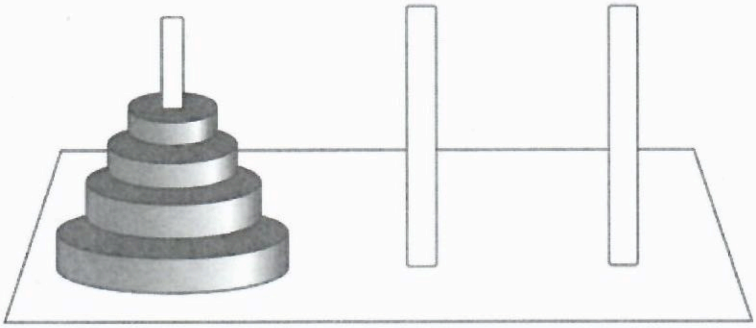
5) $(A \cap !B) \cup (!A \cap !B) = !B$

6) $(!A \cap B) \cup (!A \cap !B) = !A$

Very elegant!

Problem 3: The Towers of Hanoi puzzle is described in this figure:

The famous Towers of Hanoi puzzle involves 3 pegs with n disks of varying sizes stacked in order from the largest (on the bottom) to the smallest (on the top) on 1 of the pegs. The puzzle requires that the disks end up stacked the same way on a different peg; only one disk at a time can be moved to another peg, and no disk can ever be stacked on top of a smaller disk. Informally describe a recursive algorithm to solve the Towers of Hanoi puzzle.



Here's a recursive strategy: if you have a single disk, move it to the destination peg. Otherwise, for a stack of n disks, move the $n-1$ top disks to the third peg, move the largest (bottom) disk to the destination peg, and then move the $n-1$ disks to the destination peg, on top of the largest.

a. (5 pts) Based on the recursive strategy given, write a first-order linear recurrence relation for the number of moves $M(n)$ necessary to solve the Towers of Hanoi puzzle for a stack of n disks.

$\cancel{M(1) =}$ $M(1) = 1$

$\cancel{M(n) = M(n-1) + (n-1), \; n > 1}$

$M(n) = M(n-1) + 1 + M(n-1)$ Good

$= 2M(n-1) + 1, \; n > 1$

b. (5 pts) Solve it. How many moves $M(n)$ are necessary? (You know how many moves are necessary if $n = 1$.)

$M(n) = 2M(n-1) + 1$

$c = 2, \; g(n) = 1$

$M(n) = c^{n-1} M(1) + \sum_{i=2}^{n} c^{n-i} g(i)$

$= 2^{n-1} \cdot 1 + \sum_{i=2}^{n} 2^{n-i} \, 1$

$= 2^{n-1} + \sum_{i=2}^{n} 2^{n-i} = 2^{n-1} + (2^{n-2} + 2^{n-3} + \dots + 2^{0})$

$\cancel{= 1 + (1 + 2 + 3 + \dots + n - 1)}$ $= \frac{2^n - 1}{2-1} = 2^n - 1$

$\cancel{= 1 + \frac{n(n-1)}{2}}$

Nice work

Problem 4: Draw all distinct (non-isomorphic) rooted trees with

a. (2 pts) one vertex,

✓

b. (2 pts) two vertices, and

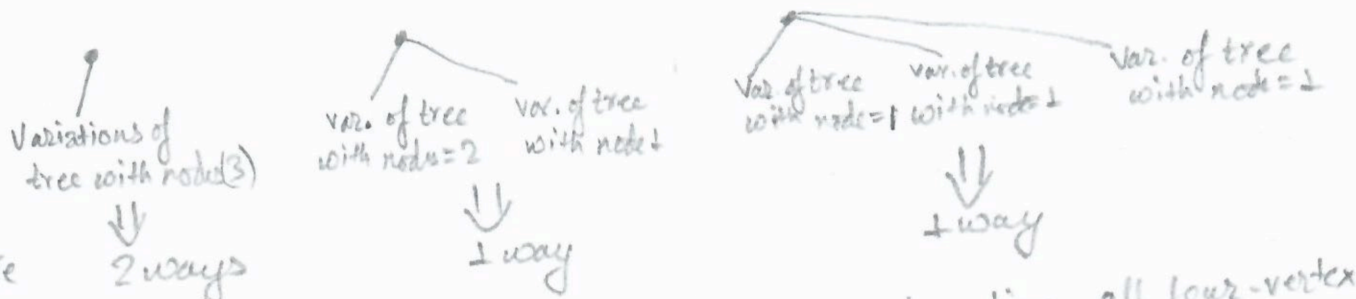
✓

c. (2 pts) three vertices;

✓

d. (4 pts) four vertices, and explain a method for recursively constructing all four-vertex rooted trees using the three previous cases.

There are



Hence, there are $(2+1+1) = 4$ ways to constructing all four-vertex rooted tree.

Well done

Problem 5: You are to prove that the proposition

$$P(n): \text{ the number of arcs in the complete graph } K_n \text{ is } A_n = \frac{n(n-1)}{2}$$

is true $\forall n \in \mathbb{N}$.

a. (4 pts) Demonstrate that the result holds for K_1 through K_4 (draw the graphs and count arcs!).

K_1 • (no arcs, $\frac{1(1-1)}{2} = \frac{1(0)}{2} = 0$)

K_2  (one arc, $\frac{2(2-1)}{2} = \frac{2(1)}{2} = 1$)

K_3  (3 arcs, $\frac{3(3-1)}{2} = \frac{3(2)}{2} = 3$)

K_4  (6 arcs, $\frac{4(4-1)}{2} = \frac{4(3)}{2} = 6$)

b. (6 pts) Use induction to prove the result. Base cases established above.

Assume $P(i)$: K_i has $\frac{i(i-1)}{2}$ arcs & consider $P(i+1)$

when you add a node to get from K_i to K_{i+1}, you have to add an arc between the $i+1^{th}$ node & each other individual node, which means you add i arcs. The # of arcs in K_{i+1} is therefore $A_i + i - 1$

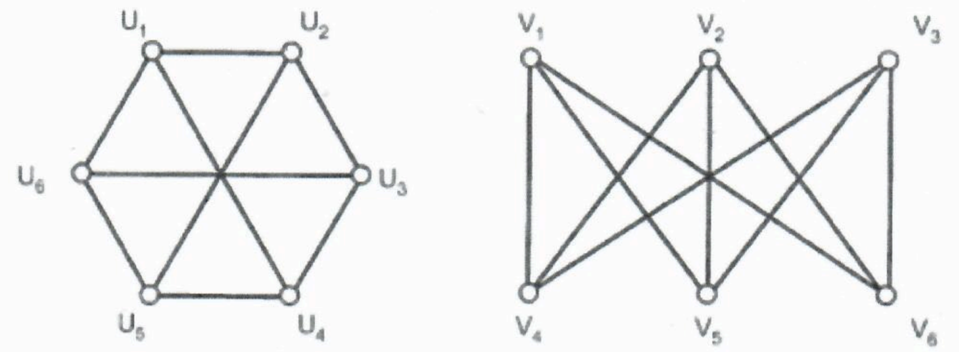
$$= \frac{i(i-1)}{2} + i = \frac{i^2 - i}{2} + \frac{2i}{2} = \frac{i^2+i}{2} = \frac{(i+1)i}{2}$$

$$= \frac{(i+1-1)(i-1)}{2}$$

Therefore, $P(n)$ is true for all $n \in \mathbb{N}$

Problem 6:

a. (5 pts) Map the nodes of one graph to the other in such a way to show that these two graphs are isomorphic:



$U_1 \to V_1$
$U_2 \to V_4$
$U_3 \to V_2$
$U_4 \to V_5$
$U_5 \to V_3$
$U_6 \to V_6$

✓

b. (2 pts) Why does it suffice to map only nodes (why don't we need a map between arcs)?

close — the graphs are simple

Arcs have a fixed start and end node, so mapping the nodes essentially also maps the start and end of any given arc ✓

What you said is true for non-simple ....

c. (3 pts) Write the adjacency matrices for the two graphs, but order the rows in the first $u_1, \ldots, u_6$, but the rows of V to correspond to your map from part a. What do you notice, and why does it make sense? (By the way, use symmetry to save effort, if possible!)

$$\begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{array} \begin{array}{c} \begin{array}{cccccc} U_1 & U_2 & U_3 & U_4 & U_5 & U_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ V_1 \\ V_4 \\ V_2 \\ V_5 \\ V_3 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_4 & V_2 & V_5 & V_3 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Mapping the nodes also maps their relations, as stated in part b. This means that the matrix, when mapped, will be the same.

The graphs are isomorphic! :)