Section Summary: 14.3

1 Definitions

The **partial derivative of f with respect to x at** (a, b) is

$$f_x(a, b) = g'(a)$$

where $g(x) = f(x, b)$. The **partial derivative of f with respect to y at** (a, b) is

$$f_y(a, b) = h'(b)$$

where $h(y) = f(a, y)$.

Hence we can define partial derivatives in terms of univariate derivatives: but, in the end, we'd rather just work in the multivariate realm, so, alternatively,

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

and

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

Then, thinking of derivative functions, rather than derivatives at a point, we define the partial derivative functions f_x and f_y by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

and

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Alternative notations (using x as the example):

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

Of course we can define partial derivatives even if we have a function of many variables: for example, if we have

$$u = f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$$

then we can define

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, x_2, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, x_2, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)}{h}$$

We can also compute higher derivatives, such as $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}$ (the second partial derivatives), and even higher order derivatives.

Partial differential equations are equations that are frequently used in physics and other sciences. For example **Laplace's equation** is a PDE,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

whose solutions describe heat distributions and are otherwise important in physics. The solutions of the **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$

have an obvious meaning!

2 Theorems

Clairaut's Theorem: Suppose f is defined on a disk D that contains the point (a, b). If the functions $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D, then

$$\frac{\partial^2 f}{\partial x \partial y}(a, b) = \frac{\partial^2 f}{\partial y \partial x}(a, b)$$

3 Properties, tips, etc.

Rule for finding partial derivatives of $z = f(x, y)$:

a. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

b. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

4 Summary

In the example of the Cobb-Douglas production function, notice that the partial derivatives are called "marginal rates": the idea is that "along the margin" one of the variables is fixed.

This is the fundamental idea of partial derivatives: that one fixes all independent variables except for the independent variable of interest, then one treats the function as an ordinary univariate function! How mundane.... Once again, univariate ideas to the rescue.

In the bivariate case, one can also think of slicing a function along lines parallel to the coordinate axes: again, the cross sections are really univariate functions, which we can differentiate as we have always done.

Clairaut's Theorem tells us that, in many cases, continuity of second partials implies that mixed partials are the same: that's one less thing for us to have to calculate if we got to partial derivatives of higher order (we get one second partial for free!).

Partial derivatives are used in partial differential equations, which are the equations which govern many phenomena in nature.