Readers may enjoy showing that these give all such pairs (details available from the author), and investigating the more general case, e.g.,

$$3x^2 + 29x + 70 = (3x + 14)(x + 5)$$
$$3x^2 + 29x - 70 = (3x + 35)(x - 2).$$

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## Linear Transformation of the Unit Circle in $R^2$

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While studying linear transformations in $R^2$, it is customary to use the image of the unit square to illustrate the effect of the transformation and the relation between its determinant and the area of the image. We show that looking at the image of the unit circle yields an appealing and informative picture and also illustrates several basic ideas.

An invertible linear transformation always maps the unit circle $U$ onto an ellipse. Suppose $T$ is an invertible linear transformation with matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} \cos t \\ \sin t \end{pmatrix},$$

then

$$\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = A^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} dx - by \\ -cx + ay \end{pmatrix}.$$

The Pythagorean identity leads to the equation of the image of $U$,

$$\left(d^2 + c^2\right) x^2 - 2(ac + bd)xy + \left(a^2 + b^2\right) y^2 = (ad - bc)^2, \tag{1}$$

an ellipse centered at the origin. However, unless $ac + bc = 0$ its axes have been rotated away from the coordinate axes. To put (1) in standard form, we diagonalize the symmetric matrix of the quadratic form. Let

$$B = \begin{pmatrix} c^2 + d^2 & -(ac + bd) \\ -(ac + bd) & a^2 + b^2 \end{pmatrix} = (\det A)^2 \left(A^{-1}\right)^t \left(A^{-1}\right)$$

so $\det B = (\det A)^2 = (ad - bc)^2 = \lambda_1 \lambda_2$, where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $B$. If $P$ is the matrix whose columns consist of the corresponding orthonormal eigenventors, let $\begin{pmatrix} x' \\ y' \end{pmatrix} = P^{-1} \begin{pmatrix} x \\ y \end{pmatrix}$. The equation of the ellipse may now be written as $\frac{x'^2}{\lambda_1} + \frac{y'^2}{\lambda_2} = 1$ and its area is $\pi \sqrt{\lambda_1 \lambda_2} = \pi\, |ad - bc| = |\det A| \times$ (area of the unit disk).

For example, if $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$, then $\det A = -6$, $T(x, y) = (2x + 2y, 2x - y)$, and the equation of $TU$ is $5x^2 - 4xy + 8y^2 = 36$. So, $B = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$ has eigenvalues 4 and 9 with corresponding eigenvectors $(2/\sqrt{5} \;\; 1/\sqrt{5})^t$ and $(-1/\sqrt{5} \;\; 2/\sqrt{5})^t$.

 

Thus $P = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and if $\begin{pmatrix} x' \\ y' \end{pmatrix} = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$, the equation becomes $4x'^2 + 9y'^2 = 36$, an ellipse with area $6\pi$. The $x'y'$ axes make an angle of $\cos^{-1}(2/\sqrt{5})$ with the $xy$ axes. The position vectors $\pm 2(2/\sqrt{5} \quad 1/\sqrt{5})^t$ and $\pm 3(-1/\sqrt{5} \quad 2/\sqrt{5})^t$ terminate at the major and minor vertices of the ellipse $TU$.

If $A = \begin{pmatrix} .5 & 0 \\ .5 & 1 \end{pmatrix}$, then $\det A = .5$ and the equation of $TU$ is $5x^2 - 2xy + y^2 = 1$. Details are left to the reader. Figures 1 and 2 show the relation of $U$ to $TU$ for the two examples.

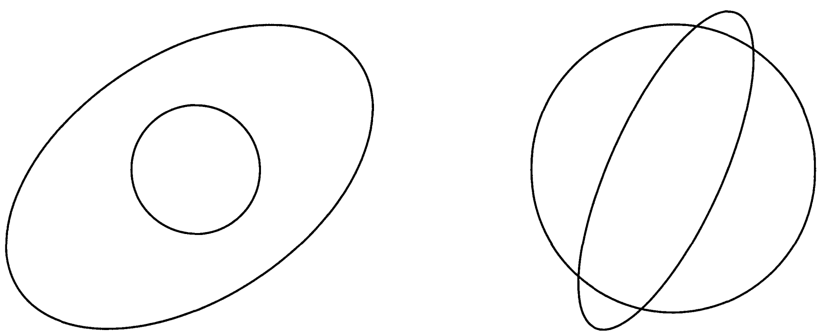

**Figure 1.**

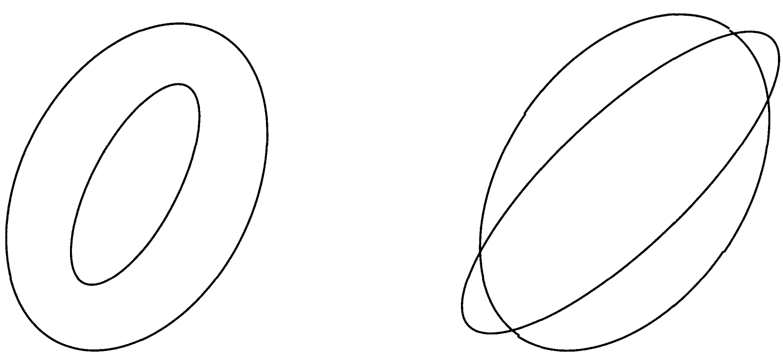

**Figure 2.**

Motivated by Schoenfeld's idea [**2**] of illustrating the vector sums $\mathbf{u} + T\mathbf{u}$ in order to identify eigenvectors of $T$ as the special vectors $\mathbf{u}$ where $\mathbf{u}$ and $T\mathbf{u}$ are parallel, Zizler and Fraser showed [**3**] that $(I + T)U$ is also an ellipse centered at the origin, though not necessarily with the same axes of symmetry as $TU$. We will now give a necessary and sufficient condition for the ellipses to have the same axes in the non-degenerate case where $T$ and $I + T$ are both invertible. If $A$ is the matrix of $T$, the condition is that

$$A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

To prove this, if we use the same technique as before with $I + A$ in place of $A$, we see that the equation of $(I + T)U$ is

$$\left((1 + d^2) + c^2\right) x^2 - 2(b + bd + c + ca)xy + \left((1 + a)^2 + b^2\right) y^2 = (1 + a + d + ad - bc)^2 \qquad (2)$$

which can be reduced to diagonal form by rotation of axes. The rotations needed to diagonalize $TU$ and $(I + T)U$ will be the same exactly when the two symmetric matrices $B$ and $C$ of the quadratic forms (1) and (2) are simultaneously diagonalizable. Since they are symmetric, they are simultaneously diagonalizable if and only if they commute [**1**]. Since $B$ and $C$ are scalar multiples of $(A^{-1})^t(A^{-1})$ and $((I + A)^{-1})^t(I + A)^{-1}$ respectively, they commute when $AA^t$ and $A + A^t$ commute, which happens exactly when $AA^t(A + A^t)$ is symmetric. Comparing the off-diagonal entries of $AA^t(A + A^t)$ we see that it is symmetric if and only if $(c - b)((a - d)^2 + (b + c)^2) = 0$. Thus $B$ and $C$ are simultaneously diagonalizable if and only if either $b = c$ or $a = d$ and $c = -b$. When this occurs, the same matrix diagonalizes both $B$ and $C$.

If $A$ is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ then $A$ is a scalar multiple of an orthogonal matrix. In this case it is easy to see that both $TU$ and $(I + T)U$ are circles with radii $\det A$ and $\det(I + A)$ respectively.

The examples used before illustrate the geometry. $A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix}$ is symmetric so the axes of $TU$ and $(I + T)U$ are the same (Fig. 2a) but $A = \begin{pmatrix} .5 & 0 \\ .5 & 1 \end{pmatrix}$ is not one of the proper forms and the axes of $TU$ and $(I + T)U$ are different (Fig. 2b).

## References


1. Stephen Friedberg, Arnold Insel, and Lawrence Spence, *Linear Algebra* 2nd ed., Prentice-Hall, 1989.
2. Steven Schoenfeld, Eigenpictures, *College Mathematics Journal* 26:4 (1995) 316–319.
3. Peter Zizler and Holly Fraser, Eigenpictures and singular values of a matrix, *College Mathematics Journal* 28:1 (1997) 59–62.


## Convergence-Divergence of $p$-Series


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There are many arguments to show that the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, including two recent proofs by contradiction by Ecker [**3**] and by Cusumano [**2**] (see also Cohen and Knight [**1**]). But the more general problem of determining the convergence or divergence of the $p$-series $\sum_{n=1}^{\infty} 1/n^p$ is almost always solved by using some form of the integral test. Here, using methods inspired by [**2**], we show how the problem can be solved without using the integral test.