MAT112 Section Summary: 7.4

The Fundamental Theorem of Calculus

Summary

This is it! We make the connection between integral and differential calculus. The slope problem (differential calculus) and the area problem (integral calculus) come together in this theorem:

Fundamental Theorem of Calculus: Let f be continuous on the interval $[a, b]$, and let F be any antiderivative of f. Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Initially we don't see the tie-in between the two: the definite integral was defined as an area; F is defined as an antiderivative of f. Why are they connected in this way? We can show that an area function is an antiderivative of f as follows: consider A(x) to equal the area under the curve from $x = a$. Hence, $A(a) = 0$.

$$A(x + h) - A(x) \approx f(x)h$$

or

$$\frac{A(x + h) - A(x)}{h} \approx f(x)$$

In the limit as $h \to 0$,

$$\lim_{h \to 0} \frac{A(x + h) - A(x)}{h} = f(x)$$

which says that A is an antiderivative of f. Hence, the total area from a to b is given by

$$\int_a^b f(x)dx = A(b) = A(b) - A(a)$$

(since $A(a) = 0$). Now, because the anti-derivatives differ by a constant only, we could have used any antiderivative in place of the area function we defined:

$$\int_a^b f(x)dx = F(b) - F(a)$$

and *voila*!

Think in terms of areas when you can, but realize that integrals can be negative! We're just adding up products $f(x) \cdot dx$, and if either $f(x)$ or dx is negative, but not both, you'll get a negative contribution.

Properties of definite integrals (think in terms of areas, and hopefully they'll make sense):

1. $$\int_a^a f(x)dx = 0$$

2. $$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$
   for any real constant k.

3. $$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$$

4. $$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$
   for any real number c; and

5. $$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Note: if you do a substitution in order to compute a definite integral, then you have to remember to change the limits of integration to reflect the new variable. The limits of integration represent the values that the differential (dx, often) is stepping off along the axis of the independent variable (usually x).

Examples: #3, 24, p. 424

Change of variable: #14, 30, p. 424

Story problem: #54, p. 425