Knots: a handout for mathcircles

Mladen Bestvina

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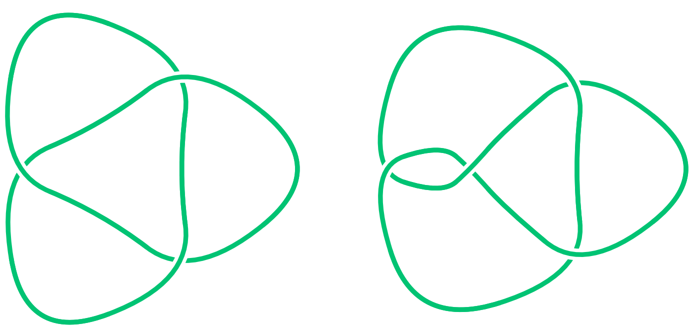
1 Knots

Informally, a *knot* is a knotted loop of string. You can create one easily enough in one of the following ways:

- Take an extension cord, tie a knot in it, and then plug one end into the other.
- Let your cat play with a ball of yarn for a while. Then find the two ends (good luck!) and tie them together. This is usually a very complicated knot.
- Draw a diagram such as those pictured below. Such a diagram is a called a *knot diagram* or a *knot projection*.



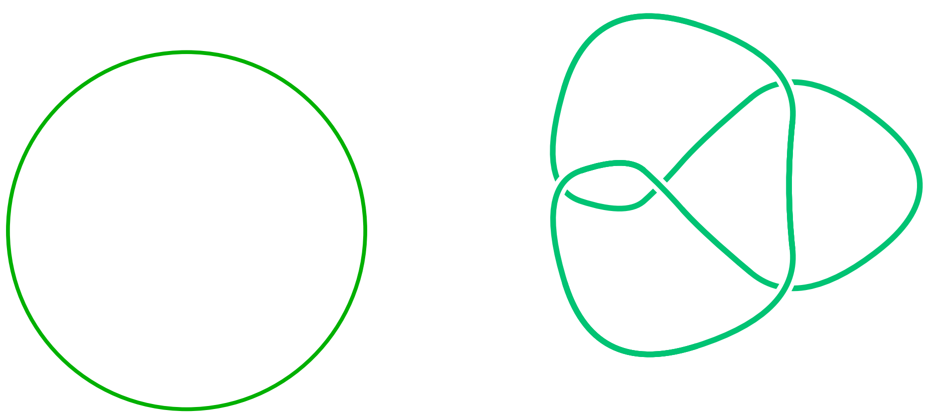
Trefoil and the figure 8 knot

The above two knots are the world's simplest knots. At the end of the handout you can see many more pictures of knots (from Robert Scharein's web site). The same picture contains many links as well. A *link* consists of several loops of string. Some links are so famous that they have names. For example, 2^2_1 is the *Hopf link*, 5^2_1 is the *Whitehead link*, and 6^3_2 are the *Borromean rings*. They have the feature that individual strings (or *components* in mathematical parlance) are untangled (or *unknotted*) but you can't pull the strings apart without cutting.

A bit of terminology: A *crossing* is a place where the knot crosses itself. The first number in knot's "name" is the number of crossings. Can you figure out the meaning of the other number(s)?

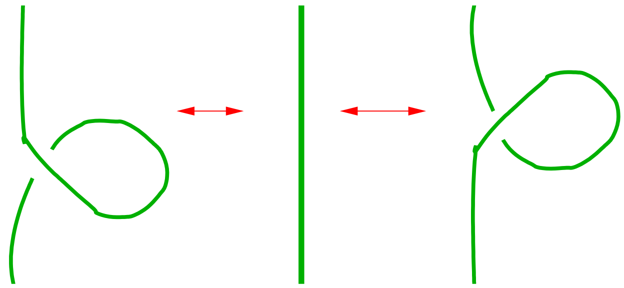
2 Reidemeister moves

There are many knot diagrams representing the same knot. For example, both diagrams below represent the *unknot*.

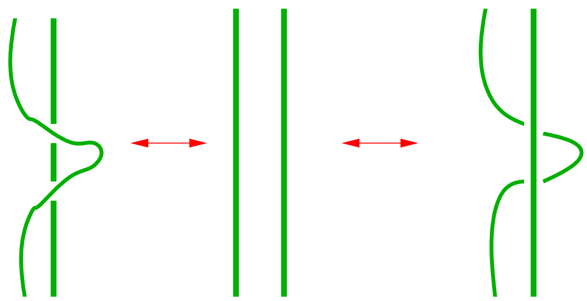


Two projections of the unknot

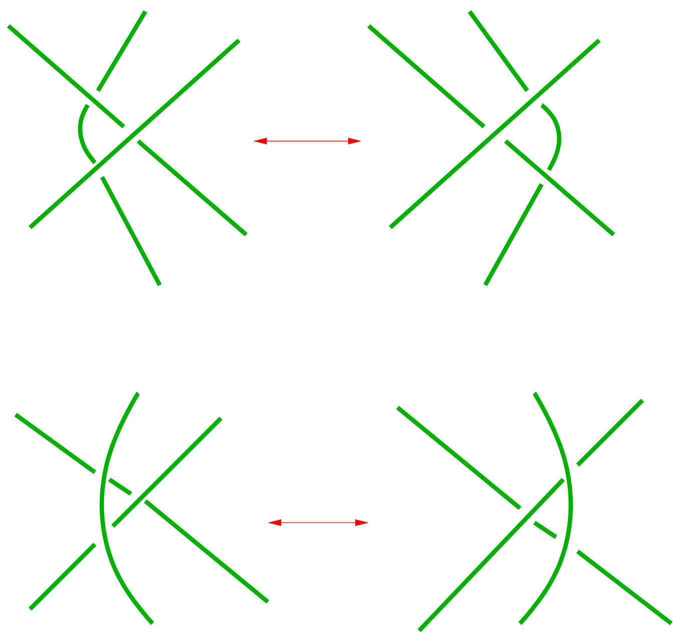
In fact, convince yourself that any of the following moves on a knot diagram will not change the knot it represents.



Type I Reidemeister move



Type II Reidemeister move



Type III Reidemeister move

1. Start with the knot diagram for the trefoil and change one of the crossings (i.e. make the upper strand go under the other one). Show that this new knot is an unknot by finding a sequence of Reidemeister moves that transforms it to a round circle.

2. Do the same with the figure 8 knot diagram.

3. Start with a round circle and then perform 4 Reidemeister moves in order to make the knot diagram more complicated. Then hand the picture to a friend to untangle it by finding simplifying Reidemeister moves.

There is a famous theorem proved by the German mathematician Kurt Reidemeister in 1920's that says:

If two knot diagrams represent the same knot (or a link) then one can be transformed to the other by a sequence of Reidemeister moves.

The catch here is that we don't know in advance how many moves will be needed.

4. The *mirror image* of a knot is obtained by reversing all crossings. Show that the mirror image of the figure 8 knot is the same knot as the figure 8 knot. It takes 8 Reidemeister moves to see this. You may want to experiment with a piece of string first.

It turns out that the mirror image of the trefoil is different from the trefoil.

Knot theory tries to answer questions such as:

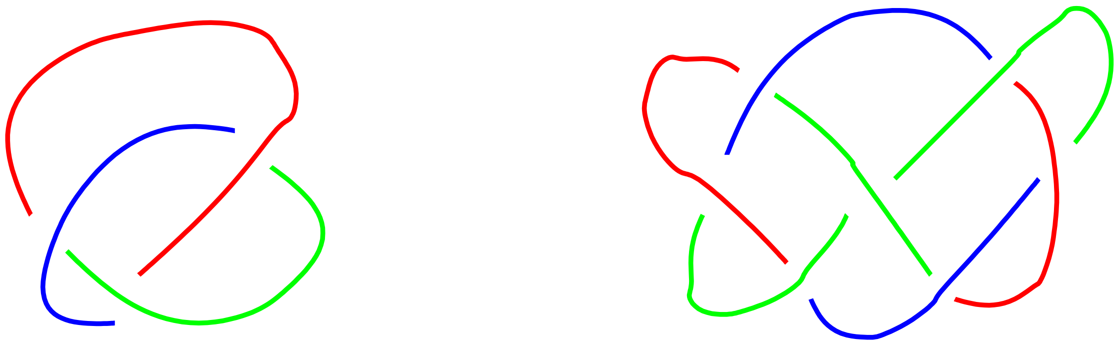
How to tell knots apart? How can we be sure that say the trefoil is really knotted and that there is no sequence of Reidemeister moves that transforms it to the unknot? Are the trefoil and the figure 8 knot really different? Can you pull Borromean rings apart without breaking them?

3 Tricolorability

A *strand* in a knot diagram is a continuous piece that goes from one under-crossing to the next. The number of strands is the same as the number of crossings.

A knot (or a link) is *tricolorable* if each strand can be colored in one of three colors with the following rules:

- At least two colors are used.
- At each crossing, either all three colors are present or only one color is present.



Trefoil and 7_4 are tricolorable

5. Decide which of the following are tricolorable: unknot, figure 8 knot, 2-component unlink, Hopf link, Whitehead link, 3-component unlink, Borromean rings. Which knots with 5, 6, 7 crossings?

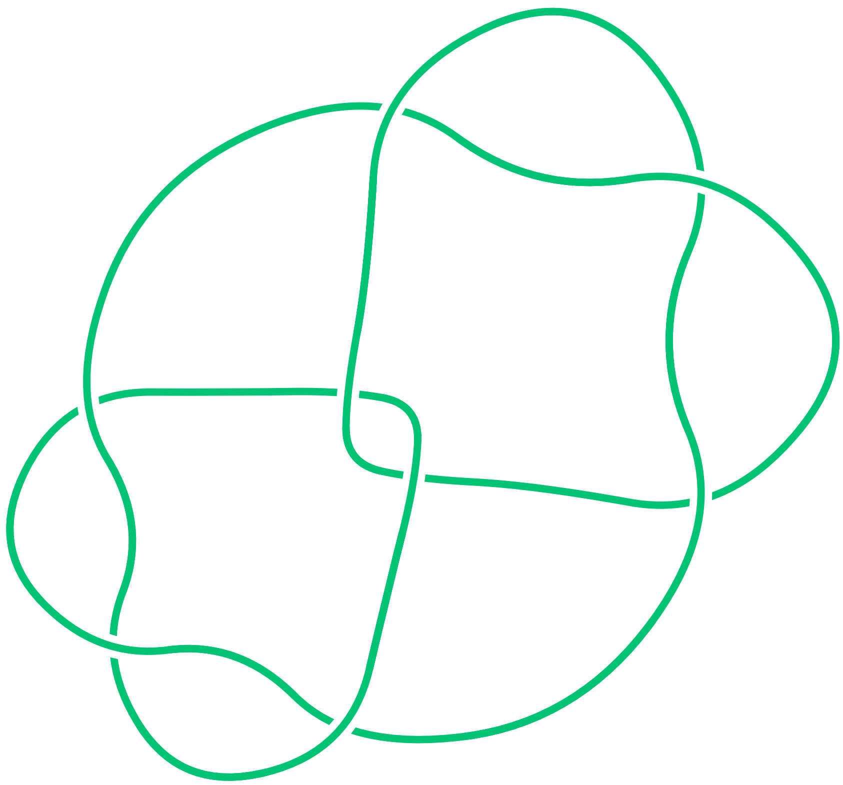
6. Show that if you start with a tricolorable knot diagram and you perform a Reidemeister move, the new knot diagram is also tricolorable.

Conclude the following:

Some knots are tricolorable and some are not, but to find out it is enough to check a single knot diagram for this knot.

7. Show that the trefoil is really a knot. Also show that the figure 8 knot is different from the trefoil, that the Hopf and Whitehead links cannot be pulled apart, and that Borromean rings cannot be pulled apart.

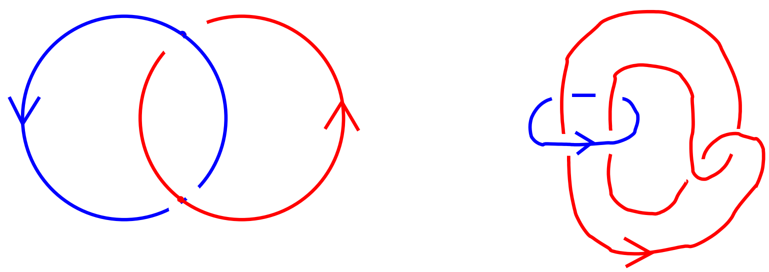
8. Show that "true lover's knot" is tricolorable (Happy Valentine's Day!).



True lover's knot

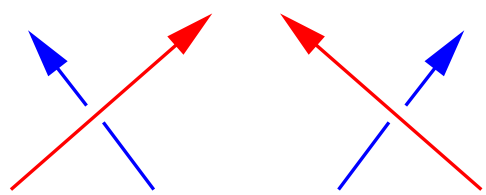
4 The linking number

Now let's think about 2-component links. We will color one componet red and the other blue. We will also choose a sense of traversing each string (an *orientation* in the parlance of knot theory).



Hopf and Whitehead links

The idea is that we want to measure how many times one component "wraps around" the other. This is called the *linking number* and can be computed as follows. Look for those crossings where the red string is above the blue string. To each such crossing assign either a $+1$ or a -1 according to the right-hand rule ($+1$ if you can place the thumb of your right hand along the red string so that the other fingers point along the blue string; otherwise -1).



Right-handed rule

9. Find a sequence of Reidemeister moves showing that the above picture of the Whitehead link and 5_1^2 represent the same link.

10. Compute the linking number for the unlink of 2 components, for the Hopf link, and for the Whitehead link.

11. What happens to the linking number if we reverse the orientation of one of the components?

12. What happens to the linking number if we perform a Reidemeister move?

13. Conclude that the Whitehead and Hopf links are really different.

14. What happens to the linking number if we switch the colors? Hint: Look at it from behind.

15. Examine the list of 2-component links at the end of the handout. Using the linking number and tricolorability, how many can you tell apart? For example 8^2_1 and 6^2_3 have different linking numbers.

16. Use linking numbers to show that 7^3_1 and 8^3_2 are different links. Find other pairs of 3-component links that you can tell apart.

5 Resources

- Colin C. Adams: *The knot book*, W.H. Freeman and Company, New York, 1999

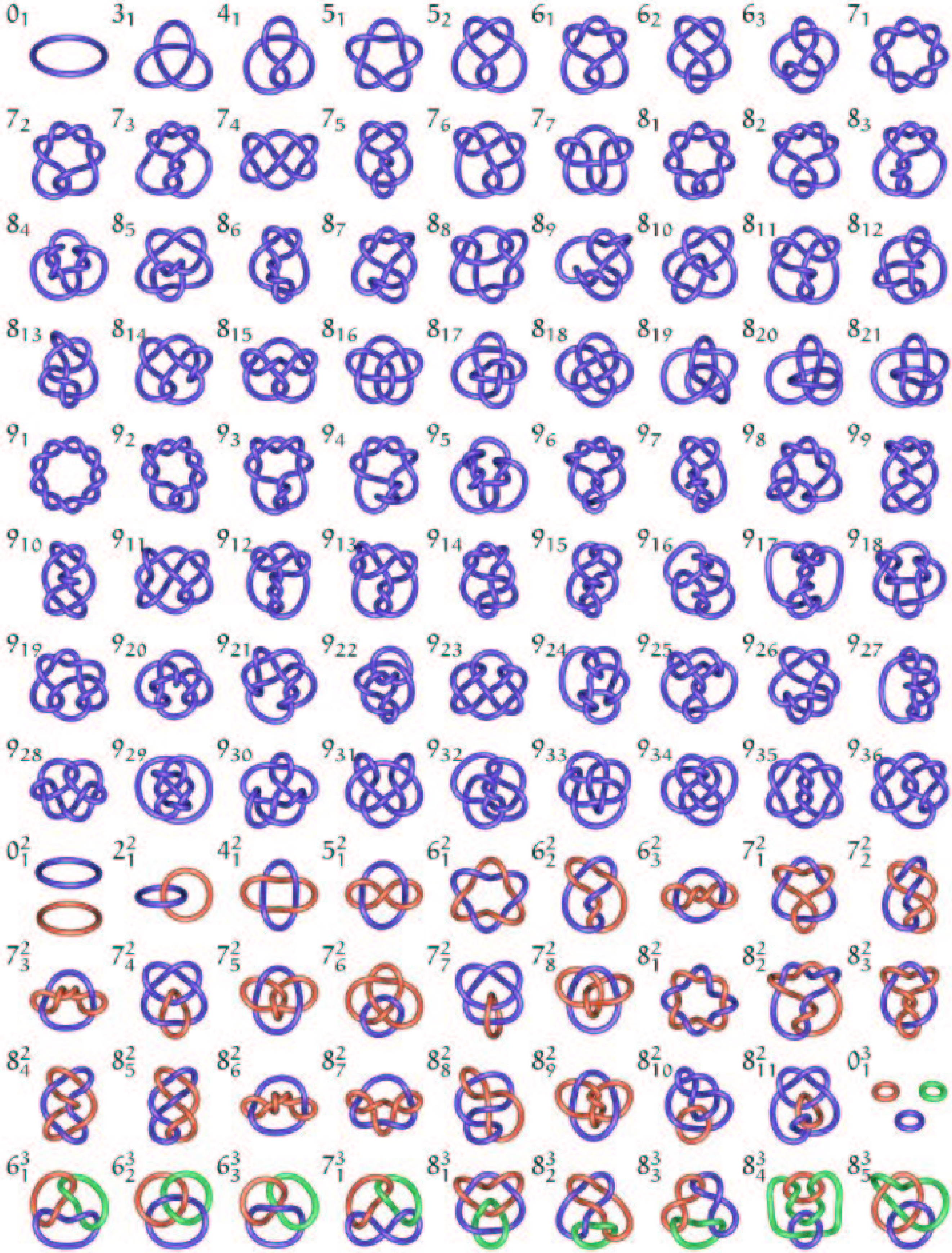
  Most topics we discussed and many more are in this very accessible book. Pick up a copy and have fun!

- `http://www.cs.ubc.ca/nest/imager/contributions/scharein/KnotPlot.html`

  Robert Scharein's excellent web site, packed with cool pictures, movies, and further links (no pun intended!). He is the author of *knotplot*, software that produces such pictures. You can download the program for free.

- `http://www.math.utk.edu/∼morwen/knotscape.html`

  Morwen Thistlethwaite's program that computes various knot polynomials. You draw a knot with the mouse and it computes the polynomials.



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