Section Summary: 14.2
Limits and Continuity

a. Definitions

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of** $f(x, y)$ **as** (x, y) **approaches** (a, b) **is** L, and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if, for every number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

whenever $(x, y) \in D$ and

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

In plain English, what this says is that when we approach (a, b) from any direction, we find ourselves getting closer and closer to L.

A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

A **polynomial function of two variables** is a sum of terms of the form $cx^m y^n$, where c is a constant and m and n are non-negative integers. A **rational function** is a ratio of polynomials.

b. Theorems

All polynomials are continuous on $\Re^2$. All rational functions are continuous on their domains. More generally, sums and products of continuous functions are continuous on their domains.

If we approach (a, b) along two different curves, and get two different limits, then the limit

$$\lim_{(x,y)\to(a,b)} f(x, y)$$

fails to exist.

c. Properties/Tricks/Hints/Etc.

We see the generalization, from approaching along a 1-dimensional ball (from left and right) in the univariate case, to approaching along a 2-dimensional ball (a circle) in the bi-variate case; in generally, we approach a point along an n-dimensional ball.

Typical problems in the univariate case are

- holes
- step functions

The same kinds of problems exist for multivariate functions, although as you can imagine by thinking of terrain, they can be generalized in many different ways.

d. Summary

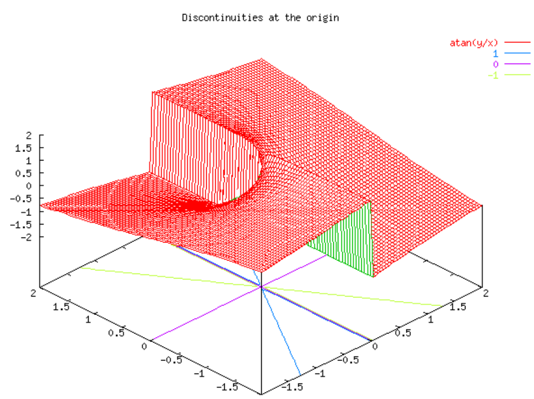
The results in this section are a simple generalization of the results from the univariate case. The idea of limit is still the same: the primary difference is that, whereas in the univariate case we were only concerned about limits from the left and from the right, now we are concerned with limits from any direction (and that's 360 degrees)!

It's nice when we can invoke limit and continuity laws of functions, which works exactly as it did in days past (sum of continuous function is continuous on the mutual domain, etc.). In particular, polynomials, rational functions, power functions, exponential functions, and trig functions are continuous on their domains

An important special case is that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

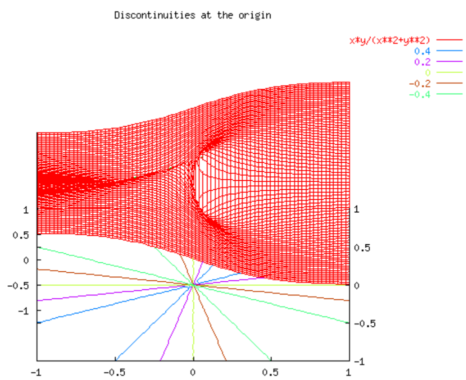
Contour plots can show us a lot about discontinuities, as we can see in the following three images:

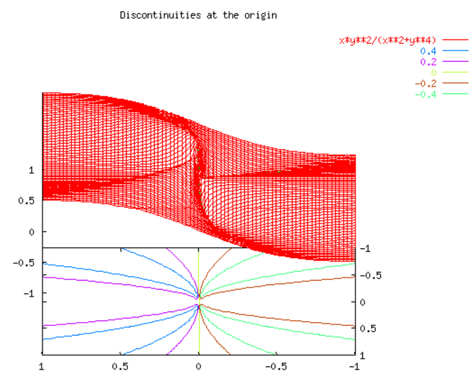
In the first function we can see gnuplot's valiant efforts to pave over the discontinuity. It really does look like a cliff in this case:



The next two functions are rational functions, which can only fail to be continuous at points not on their domains:

As is true in the case of topographic maps, when level curves collide, there's a sharp drop off (a "cliff"): cliffs mean that limits don't exist, as if you approach from certain directions you hit a "wall", rather than making a smooth transition.


Discontinuities at the origin
x*y/(x**2+y**2)
0.4
0.2
0
-0.2
-0.4


Discontinuities at the origin
x*y**2/(x**2+y**4)
0.4
0.2
0
-0.2
-0.4