Section Summary: 14.5
The Chain Rule

a. Definitions

The **dependent** variable is a function of the **intermediate** variables; these in turn are functions of the **independent** variables. Hence, the dependent variable is implicitly a function of the independent variables, but the dependence is hidden by the intermediates. The chain rule makes the dependence explicit!

b. Theorems

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are differentiable functions of t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

or

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The generalizations to multiple intermediates and multiple independent variables are relatively obvious.

c. Properties/Tricks/Hints/Etc.

Implicit differentiation is essentially an application of the chain rule. A function z is given implicitly as a function of x and y if there is there is an equation $F(x, y, z) = 0$. If you like, the surface of the function $z = f(x, y)$ is a level surface (equal to 0) of the function of three variables F!

If

- F is defined in a ball containing (a, b, c),
- $F_z(a, b, c) \neq 0$, and
- F_x, F_y, and F_z are continuous inside the sphere,

then $F(x, y, z) = 0$ defines z as a function of x and y in the vicinity of (a, b, c), and the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are defined at (a, b) to be

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = \frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

d. Summary

The chain rule in multivariate functions is a straightforward generalization of the univariate chain rule: it just requires a lot more bookkeeping to keep track of things. Tree diagrams can help us keep our books.

Implicit derivatives are really just an application of the chain rule.