Section Summary: 14.7
Maximum and Minimum Values

a. **Definitions** A bivariate function has a **local maximum** at (a, b) if

$f(x, y) \leq f(a, b)$ when (x, y) is near (a, b) (on some disk with (a, b) at its center). $f(a, b)$ is called the **local maximum value**. If $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b) (on some disk with (a, b) at its center), then $f(a, b)$ is called a **local minimum value**.

If the inequalities above hold for the entire domain, then the extrema are **absolute**.

(a, b) is a **critical point** (or **stationary point**) of f if $f_x(a, b) = f_y(a, b) = 0$, or if either partial doesn't exist.

A **closed set** in $\Re^2$ is one which contains its boundary (for example, a circle and its interior; or any standard geometric area and its perimeter).

b. **Theorems** If f has a local maximum or minimum at (a, b) and the

first-order partial derivatives of f exist there, then $f_x(a, b) = f_y(a, b) = 0$.

If the second partials are continuous on a disk with center (a, b), a critical point, then define

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

There are three cases:

i. If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
ii. If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
iii. If $D < 0$, then $f(a, b)$ is neither a local maximum nor minimum (in this case, (a, b) is called a **saddle point**).

Note the weird asymmetry: why do we focus on f_{xx}, rather than f_{yy}? Take a look at the conditions, and you'll see that, in the first case, if $D > 0$ and $f_{xx}(a, b) > 0$ then we can conclude that $f_{yy}(a, b) > 0$ is also true; similarly, in the second case, if $D > 0$ and $f_{xx}(a, b) < 0$ then $f_{yy}(a, b) < 0$ is also true.

So this could be rephrased to eliminate the asymetry, which might be a good idea....

Extreme Value Theorem for bivariate functions: If f is continuous on a closed, bounded set R in $\Re^2$, then f attains absolute extrema on R.

c. Properties/Tricks/Hints/Etc.

To find the absolute extrema of a continuous function f on a closed, bounded set S:

i. Find the values of f at the critical points of f in S.
ii. Find the extreme values of f on the boundary of S.
iii. The absolute extrema are the maximum and minimum values from steps 1 and 2.

d. Summary

Okay! It's time to optimize. The objective is to find those maxes and mins, and we're going to do it in the same old way: finding critical values, and checking on the boundaries. Even the second derivative test has its analogy (at least in the bivariate case).

Even the strategy for finding absolute extrema is the same.