Section Summary: 15.8

Triple Integrals in Cylindrical Coordinates

a. Definitions

Cylindrical coordinates represents one generalization of polar coordinates:

$x = r\cos\theta$ $\quad y = r\sin\theta$ $\quad r^2 = x^2 + y^2$ $\quad z = z$

So it represents a polar coordinate transformation for two coordinates, and leaves z untransformed.

This makes sense, since an infinite cylinder centered at the origin with radius r is described by $C = \left\{(x, y, z) | r^2 = x^2 + y^2\right\}$ (and z is unconstrained). (Figure 4, p. 1052)

b. Theorems

Writing a rectangular integral as a polar integral:

$$\iiint_R f(x, y)\mathrm{d}A = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\cos(\theta))}^{u_1(r\cos(\theta), r\cos(\theta))} f(r\cos(\theta), r\sin(\theta), z) r dz \mathrm{d}r \mathrm{d}\theta$$

The volume element ($\mathrm{d}V$) is given by $rdz\mathrm{d}r\mathrm{d}\theta$, as one can see in Figure 7, p. 1054. This is a simple generalization of the polar formula of the area element $\mathrm{d}A$.

c. Properties/Tricks/Hints/Etc.

The cone is has a beautiful representation in cylindrical coordinates: $z = ar$, where a is a constant. See Figure 5, p. 1053, where $a = 1$.

d. Summary

This is a means to computing integrals with certain geometries more simply (e.g. cylinders and cones).

Coordinate transformations are a common practice in mathematics: if it's up to you, choose the coordinates and the coordinate system that make your work as easy as possible.