Section Summary: 15.8
Triple Integrals in Spherical Coordinates

a. Definitions

Spherical coordinates (ρ, θ, ϕ) are perfect for a sphere, unsurprisingly. A spherical surface of radius r is given by

$$\rho = r$$

That's a really simple formula for a fairly complex object. But that's because the coordinate system is built for the object!

$$\theta = \theta_0$$

is a plane, through the origin, at an angle of θ_0 from the x-axis, passing through the z-axis. It slices the sphere in a great circle, through the north and south poles. Obviously a very important slice (longitude lines) through a sphere such as the Earth (at least approximately spherical), where satellites pass over great circles, often circumnavigating the globe over the poles.

Latitude lines (of constant ρ)) are also very important, and are given by the simple formulas

$$\phi = \phi_0$$

This is gives rise to a circle of radius $\rho \sin(\phi_0)$, perpendicular to the z-axis, which runs through its center. If we allow ρ to vary, from 0 to infinity, then we trace out a cone.

See Figure 5, p. 1057.

b. Theorems

If E is a "spherical parallelpiped" (generalization of a "polar rectangle",

$$E = \{(\rho, \theta, \phi) | a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

then

$$\int\int_E\int f(x, y, z) dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))\ \rho^2 \sin(\phi)\ d\rho\ d\theta\ d\phi$$

c. Properties/Tricks/Hints/Etc.

The element of volume dV for cylindrical coordinates is a simple generalization of that for polar coordinates:

$$r\ dr\ d\theta \longrightarrow r\ dr\ d\theta\ dz$$

whereas for spherical coordinates,

$$dV \longrightarrow \rho^2\ \sin(\phi)\ d\rho\ d\theta\ d\phi$$

Notice that the volume element has dimensions "length squared": ρ^2 multiplies $d\rho$. **Units are important and often a very helpful guide or check.**

d. Summary

Changing coordinate systems is not done frivolously, but generally because of some symmetry properties of a problem that make the problem easier to represent (or to solve) in that system.

Coordinate systems are “known” (in some sense) by the equations obtained by setting each coordinate variable equal to a constant. In rectangular coordinates, mutually orthogonal planes result. In the case of cylindrical coordinates, the three surfaces are two mutually orthogonal planes and a cylinder. Where they intersect in a point, the three surfaces are (locally) mutually orthogonal.

For spherical coordinates, the three surfaces are a plane, a spherical shell, and a cone: again, at their intersection in a point, they are locally mutually orthogonal.

The important upshot is that these spherical coordinates are ideal for representing an ice cream cone, which is a tremendously important applied problem, which should probably be studied often (perhaps with varying flavors of ice cream, which have different density functions).