Section Summary: 16.2
Line Integrals

a. Definitions

If f is defined on a smooth curve C given by $(x(t), y(t))$, $a \leq t \leq b$, then the **line integral of f along C** is

$$\int_C f(x, y)ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s$$

If f is continuous, then

$$\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}dt$$

and this is independent of the parameterization (provided the curve is traversed just once) as t increases from a to b.

This generalizes easily to curves in space:

$$\int_C f(x, y, z)ds = \int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

line integrals of f along C with respect to x and y:

$$\int_C f(x, y)dx = \int_a^b f(x(t), y(t))x'(t)dt$$

and

$$\int_C f(x, y)dy = \int_a^b f(x(t), y(t))y'(t)dt$$

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of $\mathbf{F}$ along C** is

$$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_C \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

where $\mathbf{T}$ is the tangent vector (the component of velocity in the direction tangential to the motion).

If vector field $\mathbf{F} = P\hat{\imath} + Q\hat{\jmath} + R\hat{k}$, then we can break the integral into three:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz$$

b. Theorems

In general, if we change the orientation of the parameterization (s runs from b to a), then

$$\int_{-C} f(x,y)ds = -\int_C f(x,y)ds$$

This is because the differential ds is negative (going from larger values to smaller).

When ds is considered a length (e.g. the computation of arclength), then we have the formula

$$\int_{-C} f(x,y)ds = \int_C f(x,y)ds$$

(see p. 1092). This is confusing, but recall how ds was defined:

$$ds^2 = dx^2 + dy^2$$

then

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, dt$$

where we think of dt as positive. It would probably be better to write

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\, |dt|$$

but we don't. We're lazy mathematicians! And that causes you problems.... I apologize on behalf of mathematicians anywhere, but am unable to make the big changes necessary.

c. Properties/Tricks/Hints/Etc.

If the curve C is made up of a bunch of smooth curves C_i, then the line integral over C is just the sum of the line integrals over the C_i.

d. Summary

We've seen line integrals before, in the context of the calculation of arc length back in our univariate calculus days. The first problem is to parameterize the curve along which you're integrating. Once we have our parameterization, the integral reduces to an ordinary integral of a single variable (usually called t), the variable of the parameterization. Today with a GPS unit and a watch it's easy to parameterize a curve in space.

Then we simply multiply the "height" $(f(x(t), y(t)))$ times the tiny chunk of arc length $(ds(t))$ to get our integral. We can do this for a curve embedded in the plane, or for a curve in space: the difference is simply one of additional labor.

One of the twists here is that we're going to be computing line integrals with vector fields (e.g. calculating "work", in its technical sense of force through distance). It's not a big deal: the scalar function we're working with in that case is simply the dot product of the force and displacement vectors.