Section Summary: 16.3
Fundamental Theorem for Line Integrals

a. Definitions

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is **independent of path** if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

for any two paths C_1 and C_2 in D that have the same initial and terminal points.

A curve is called **closed** if its terminal point coincides with its initial point. It's **simple** if it doesn't intersect with itself between endpoints.

A region D is **open** if about every point P in D there is a disk with center P lying entirely within D. It is **connected** if any two points in D can be connected by a path lying entirely in D (the region is in "one piece"). It is **simply-connected** if every simple closed curve in D encloses only points that are in D (it's got no holes in it).

b. Theorems

i. Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) \tag{1}$$

ii. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

iii. If $\mathbf{F}(x, y) = P(x, y)\ \mathbf{i} + Q(x, y)\ \mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

This says that mixed partials of f (the potential of $\mathbf{F}$) are equal – the proof is Clairaut's theorem! $P = f_x$ and $Q = f_y$, so

$$f_{xy} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = f_{yx}$$

iv. Let $\mathbf{F} = P\ \mathbf{i} + Q\ \mathbf{j}$ be a vector field in an open, simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ throughout } D \tag{2}$$

Then $\mathbf{F}$ is conservative.

v. The **Law of Conservation of Energy**: that the sum of an object's potential and kinetic energies remains constant.

vi.

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of path in D if and only if

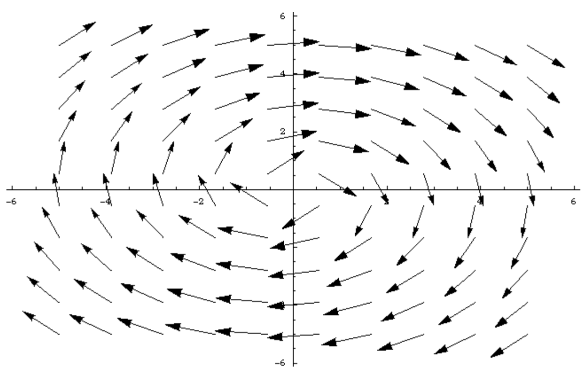
$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

for every closed path C in D.

c. Properties/Tricks/Hints/Etc.

We can use theorem (**??**) to show that a vector field is not conservative. Simply show that the indicated partials are not equal, and the field cannot possibly be conservative.

Another test is the closed path test: if you can find a closed path on which clearly the work done in traversing it is not zero, then the field is not conservative. For example,



d. Summary

The focus in this section is on conservative fields. Don't lose the focus! Conservative fields are the most naturally occurring, as they are associated with scalar functions: if you've got a scalar function defined, then usually it's differentiable, and hence there's a vector field representing a flow. So this important case gets extra attention.

It turns out that there's a variation of the fundamental theorem of calculus for line integrals representing work for conservative fields (Equation **??**).

The crowning glory of the section is the result on the conservation of energy. Realizing this result as a simple consequence of Newton's laws and integration theory is as good as it gets!