Section Summary: 16.4
Green's Theorem

a. **Definitions** A simple closed curve is **positively oriented** if it is traversed in a counter-clockwise direction. This means that the region is to the left as we traverse C.

b. **Theorems**

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C (C is sometimes denoted ∂D in this case, as the boundary of D). If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_{\partial D} P dx + Q dy = \int_D \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Pardon the re-use of the ∂ symbol in this way! The symbol $\oint$ indicates that the boundary is traversed in the positively oriented way.

c. **Properties/Tricks/Hints/Etc.**

From 16.3 (the fundamental theorem of line integrals) we discovered that for gradient (or conservative) fields $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Hence

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

which means that Green's theorem in the case of gradient fields becomes

$$\int_C P dx + Q dy = \int_C \mathbf{F} \cdot d\mathbf{r} = 0$$

We knew that!

Green's theorem can be used in a sneaky way to calculate areas: if we find functions P and Q so that

$$A = \int_D \int dA = \int_D \int 1 dA = \int_D \int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_{\partial D} P dx + Q dy$$

then we can use a line integral to compute area! Fortunately such pairs of P and Q are easy to find:

- $P = 0$, $Q = x$;
- $P = -y$, $Q = 0$;
- $P = -y/2$, $Q = x/2$

If any particular pair makes the line integral easy, then we're in clover....

d. Summary

Green's theorem is simply a calculation of a rather special integral on a two-dimensional region D:

$$\int_D\int\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA=\oint_{\partial D}Pdx+Qdy=\oint_{\partial D}\mathbf{F}\cdot d\mathbf{r}$$

It shows one way to handle the "work problem" $\int_C \mathbf{F}\cdot d\mathbf{r}$ when the field is not conservative. It can also be seen as a generalization of the Fundamental Theorem of Calculus to area integrals, in the sense that the integral defined on a region can be evaluated by considering only its boundary.

We can use this backwards, however: if the boundary calculation is ugly, it may be that the area integral is easier!

Let's show how Green's theorem is just a generalization of the Fundamental Theorem of Calculus to area integrals: consider

$$I=\int_D\int h(x,y)dA=\int_a^b\int_{g_1(x)}^{g_2(x)}h(x,y)dydx$$

for a simple region D. If we can think of a $P(x,y)$ such that

$$h(x,y)=-\frac{\partial P}{\partial y},$$

then

$$I=\int_a^b\int_{g_1(x)}^{g_2(x)}-\frac{\partial P}{\partial y}dydx=-\int_a^b\left(P(x,g_2(x))-P(x,g_1(x))\right)dx$$

Now let's consider

$$\oint_C P(x,y)dx=\int_{C_1}P(x,y)dx+\int_{C_2}P(x,y)dx$$

where C_1 and C_2 are the integrals over the bottom and top of C, respectively. Using $x, g_1(x)$ as the parameterization for C_1 and similarly for C_2, we have

$$\oint_C P(x,y)dx=\left\{\begin{array}{c}\int_a^b P(x,g_1(x))dx+\int_b^a P(x,g_2(x))dx\\ \int_a^b P(x,g_1(x))dx-\int_a^b P(x,g_2(x))dx\\ -\int_a^b\left(P(x,g_2(x))-P(x,g_1(x))\right)dx\end{array}\right\}=\int_D\int h(x,y)dA$$