Section Summary: 16.5: Curl and Divergence

a. Definitions

- The **del** operator:
$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$
We've seen ∇ before, of course: when it acts on a scalar function f, it returns the gradient of f, ∇f.
There are three ways that we use ∇:
    i. as an operator (which is a type of function, which takes functions as arguments, rather than numbers). This is exemplified by the gradient, but also by the Laplacian.
    ii. to multiply vector field functions, applying both of the two types of vector multiplication – dot- and cross-products. Using the dot-product, ∇ creates a scalar result; using the cross-product, ∇ creates a vector.

- **Curl of vector field $\mathbf{F} = \langle P, Q, R \rangle$:**
$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$
If curl $\mathbf{F} = 0$ at a point, then $\mathbf{F}$ is said to be **irrotational** at that point.

    Making use of ∇, then, curl $\mathbf{F} = \nabla \times \mathbf{F}$, where the symbol $\times$ is the cross-product.

- **Divergence**:
$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \mathbf{F}$$
If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

- The **Laplacian** operator is designated by ∇^2, and creates a scalar function from a scalar function. We can think of this as
$$\nabla^2 f = \nabla \cdot \nabla f.$$

b. Theorems

- If f is a function of three variables that has continuous second-order partial derivatives, then
$$\text{curl}(\nabla f) = \mathbf{0}$$
(note that that's a vector $\mathbf{0}$). This says that if $\mathbf{F}$ is conservative, then curl $\mathbf{F} = \mathbf{0}$.

- If $\mathbf{F}$ is a vector field defined on all of $\Re^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

- If $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$ is a vector field on $\Re^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = \mathbf{0}$$

- **Green's theorem**:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_D \int (\text{curl } \mathbf{F}) \cdot \mathbf{k} dA$$

- **Green's theorem** (normal components):

$$\oint_C \mathbf{F} \cdot \mathbf{n} ds = \int_D \int (\text{div } \mathbf{F}) dA$$

c. Properties/Tricks/Hints/Etc.

There are several important things to know about cross-products:

$$\hat{\mathbf{i}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}}$$
$$\hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}$$
$$\hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}$$

Furthermore,

$$\mathbf{u} \times \mathbf{v} = -\mathbf{u} \times \mathbf{v}$$
$$\mathbf{u} \times \mathbf{u} = \mathbf{0}$$

One important thing to know about the cross-product is that it is perpendicular to the two vectors that make it up:

$$\mathbf{u} \times \mathbf{v} \perp \mathbf{u} \qquad\qquad \mathbf{u} \times \mathbf{v} \perp \mathbf{v}$$

If curl $\mathbf{F} \neq \mathbf{0}$, then $\mathbf{F}$ is **not** conservative.

In the context of fluid flow,

i. The divergence measures the tendency of a fluid to diverge from the point (x, y, z), whereas
ii. the curl measures the rotational tendency of the fluid (about the axis given by the direction of the curl) at the point (x, y, z).

d. Summary

In this section we encounter two important extensions of the gradient operator (also known as "del"): del operates on a scalar function to produce the gradient. In addition,

- $\nabla \times \mathbf{F}$ produces a vector field called the curl of $\mathbf{F}$; and
- $\nabla \cdot \mathbf{F}$ produces a scalar field called the divergence of $\mathbf{F}$.

We discover two equivalent vector-formulation of Green's theorem which allows us to use the result in three-space, and understand it in the context of fluid flow.

The curl also provides us with a way of determining whether a vector field is conservative (that is, a gradient field).