MAT360 Section Summary: 2.2
Fixed-Point Iteration

Summary

Suppose that you want to solve the equation

$$\cos(x) = x$$

The value of x that satisfies this equation is called a **fixed point** for the function $g(x) = \cos(x)$, because it is a point such that $g(x) = x$ – the image is the same as the argument.

One way to go about finding the fixed point would be to rewrite the equation as

$$f(x) = \cos(x) - x = 0$$

and to use bisection to find a root of f (in fact, the unique root, as one can see from the graph of f).

Fixed-point iteration is based on a couple of results from calculus: the IVT, and the MVT, as follows:

Theorem 2.2:

- If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has a fixed point in $[a, b]$.
- If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with
$$|g'(x)| \leq k, \quad \forall x \in (a, b),$$
then the fixed point in $[a, b]$ is unique.

The proofs are by

- the IVT, with $h(x) = g(x) - x$; and

- the MVT, and contradiction.

So we know that there's a fixed point on an interval $[a, b]$, and may even know that it's unique. What now?

Now we assume that, perhaps, if we start with a value x_0 that's close to the real fixed point p, that by simply computing $g(x_0)$ (which is $\approx x_0$) we'll actually get closer to p.

Let's look at the "cobweb diagram" of this situation.

Under what circumstances will that happen? In what circumstances would the same "cobwebbing" procedure fail?

Well, in some circumstances, it's guaranteed to work:

Theorem 2.3: Fixed-Point Theorem Let $g \in C[a, b]$ be such that $g(x) \in [a, b]$, for all x in $[a, b]$. Suppose, in addition, that g' exists on (a, b), and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k$$

for all $x \in (a, b)$. Then for any number $p_0 \in [a, b]$, the sequence

$$p_n = g(p_{n-1})$$

$n \geq 1$, converges to the unique fixed point p in $[a, b]$.

Proof: MVT applied to $|p_n - p|$.

Corollary 2.4: If g satisfies the hypotheses of Theorem 2.3, then bounds for the error involved in using p_n to approximate p are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

for all $n \geq 1$.

Proof: by use of various inequalities.

There may be lots of ways to create a fixed-point function, and some of them are better than others.

Example: consider exercises 1 and 2.