MAT360 Section Summary: 2.3

Newton's Method

Summary So... you may be wondering what the big deal is about Fixed Point Iteration (FPI): I mean, there are all these different functions that you can use, but we don't seem to know how to guarantee that one is good and another bad: some work, others are disasters. If only we could find a function that we could be sure of....

That danged Newton! He seemed to find all the cool stuff. He did it this time, once again. It wasn't enough that he discovered the law of Universal Gravitation, the theory of colors, integral and differential calculus, the binomial theorem, Newton's law of cooling, the solution to the brachistochrone problem: he had to discover Newton's method too.... Who better, however? What's the chance that the discover would have the name Newton too?;)

The method is best approached from the direction of our old friend, the Taylor series expansion, and the root problem (the dual problem of the FP problem): write

$$f(x) = f(p_0) + f'(p_0)(x - p_0) + O((x - p_0)^2)$$

or

$$f(x) \approx f(p_0) + f'(p_0)(x - p_0)$$

When $x = p$, $f(p) = 0$: so, starting from p_0, perhaps a better estimate of p will be obtained by solving the the following equation for p_1 such that $f(p_1) = 0$:

$$0 = f(p_0) + f'(p_0)(p_1 - p_0)$$

or

$$p_1 = p_0 - \frac{f(p_0)}{f'(p_0)}$$

and, more generally,

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}$$

(provided $f'(p_{n-1}) \neq 0$). This is our scheme, illustrated in Figure 2.7, and encapsulated in the fixed point function

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Notice that $f(x) = 0 \Longrightarrow g(x) = x$ (provided $f'(x) \neq 0$).

The really neat thing about this FPI function g is that $g'(p) = 0$, which means that convergence of the FPI scheme will be quite fast:

$$g'(x) = 1 - \frac{f'(x)^2 - f(x)f''(x)}{f'(x)^2}$$

If $f(p) = 0$, then

$$g'(p) = 1 - \frac{f'(p)^2}{f'(x)^2} = 0$$

This is the feature that makes Newton's method so very grand and wonderful, and which justifies our interest in the method.

In fact, Newton's method is said to converge **quadratically**, by contrast with FPI which is generally said to converge **linearly**:

$$g(p_n) = p + g'(p)(p_n - p) + \frac{1}{2}g''(p)(p_n - p)^2 + \ldots$$

or

$$p_{n+1} = p + \frac{1}{2}g''(p)(p_n - p)^2 + \ldots$$

so that

$$|p_{n+1} - p| \approx \frac{1}{2}|g''(p)||p_n - p|^2$$

when p_n gets into close proximity to p. When will we be assured of "contracting"? When

$$\frac{1}{2}|g''(p)||p_n - p| < 1$$

Obviously, as long as $g''(p)$ is bounded, there is a neighborhood of p in which this will happen.

Problems in paradise....

What makes it not quite as good as sliced bread?

- It won't find complex roots (starting from real values), that's for sure;
- Unlike bisection, it doesn't produce iterates that bracket a root;

- Well, we have to calculate derivatives, for one thing, and those are generally expensive – also we have to be able to compute values from them, which can be computationally expensive.

One cheap fix for the derivative problem is to use the discrete approximation to the derivative: that is,

$$f'(p_{n-1}) \approx \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}$$

Hence, the iteration function becomes

$$p_n = p_{n-1} - \frac{p_{n-1} - p_{n-2}}{f(p_{n-1}) - f(p_{n-2})} f(p_{n-1})$$

which is the heart of the **secant method**, which requires two approximations to start (p_0 and p_1). One of the neat things about the secant method is that it requires only one additional function value calculation at each step (in lieu of a derivative, it requires some bookwork – remembering old iterates – and additional arithmetic – differences and quotients). The secant method doesn't converge quadratically like Newton's method (although it does converge super-linearly, with exponent ≈ 1.618. Hence, it is still better than bisection in terms of convergence; it may, however, fail to bracket a root (it will lose track of the interval on which a solution exists...).

The method of **False Position** (*Regula Falsi*) is just a modified bisection, that should approach the root faster (because it uses secant lines rather than midpoints), but has the unfortunate property that it doesn't necessarily produce a diminishing interval squeezing the root (as illustrated in Figure 2.10). While an interesting twist on the bisection method, it is not recommended.