MAT360 Section Summary: 4.4
Composite Integration Schemes

1. Summary

We can continue to generate higher-order Newton-Cotes methods, but the cost is needing to use more and more points, and increasingly complex coefficients schemes. An alternative strategy is to break the interval $[a, b]$ into the elemental patches that we used to define the lower-order methods that we studied in section 4.3, and integrate over the patches and add up the results.

2. **Definitions**

- **Composite scheme**: divide the interval $[a, b]$ into n patches appropriate to a particular method, apply that method on the patches, then add the results to approximate the integral $\int_a^b f(x)dx$.

3. **Theorems/Formulas**

Composite Trapezoidal Rule: we divide the interval $[a, b]$ into n panels, with $h = \frac{b-a}{n}$. Then we add up the estimates on all the panels. The easiest way to do this is using linear algebra. Given the values of the function at the $n + 1$ points $x_i, i \in \{0, \cdots, n\}$, the weight matrix for the calculation of a vector of individual estimates on each of the subpanels is given by

$$W = \frac{h}{2}\begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \end{bmatrix}$$

The estimate for the integral is then given by adding them all up:

$$I \approx [1\ 1\ \cdots\ 1]_{1\times n} W_{n\times n+1} \overline{f}_{n+1\times 1}$$

The product $[11\cdots 1]W$ is constant, so we can do that product once and for all:

$$I \approx \frac{h}{2}[1\ 2\ 2\ \cdots\ 2\ 1]\overline{f}$$

The pattern $[1\ 2\ 2\ \cdots\ 2\ 1]$ might be called the trapezoidal dance. We have the following theorem. In addition, we can calculate the error term by summing up the error terms for all the intervals: for the trapezoidal rule we make an "elemental error" of the form

$$-\frac{h^3 f^{(2)}(\xi_i)}{12}$$

on each panel. Hence the total error for the interval is

$$E = -\sum_{i=1}^{n} \frac{h^3 f^{(2)}(\xi_i)}{12}$$

which can be written as (see Theorem 4.5)

$$E = -\frac{(b-a)h^2 f^{(2)}(\mu)}{12}$$

by taking μ such that $f^{(2)}(\mu)$ is the mean value of the error terms (and, provided the second derivative is continuous, we can obtain the following by the mean value theorem):

$$f^{(2)}(\mu) = \frac{1}{n}\sum_{i=1}^{n} f^{(2)}(\xi_i) = \frac{h}{b-a}\sum_{i=1}^{n} f^{(2)}(\xi_i)$$

Theorem 4.5: Let $f \in C^2[a,b]$, $h = \frac{b-a}{n}$, and $x_i = a + ih$. Then $\exists \mu \in (a,b)$ for which

$$\int_a^b f(x)dx = \frac{h}{2}[1\ 2\ 2\ \cdots\ 2\ 1]\overline{f} - \frac{b-a}{12}h^2 f''(\mu)$$

The **Composite Midpoint rule** has a similar result associated with it: again we divide the interval $[a,b]$ into n panels, with $h = \frac{b-a}{n}$, then add up the estimates on all the panels. Given the values of the function at the n points $x_i + \frac{h}{2}, i \in \{0, \cdots, n-1\}$, the weight matrix for the calculation of a vector of individual estimates on each of the subpanels is just

$$W = hI_{n \times n}$$

The estimate for the integral is then given by adding them all up:

$$I \approx [1\ 1\ \cdots\ 1]_{1\times n} W_{n\times n} \overline{f}_{n\times 1}$$

to yield

$$I \approx h[1\ 1\ \cdots\ 1\ 1]\overline{f}$$

or just

$$I \approx h \cdot sum(\overline{f})$$

We again calculate the error term by summing up the error terms for all the intervals: for the midpoint rule we make an "elemental error" of the form

$$\frac{h^3 f^{(2)}(\xi_i)}{24}$$

on each panel. Hence the total error for the interval is

$$E = \sum_{i=1}^{n} \frac{h^3 f^{(2)}(\xi_i)}{24}$$

which can be written as (see Theorem 4.5)

$$E = \frac{(b-a)h^2 f^{(2)}(\mu)}{24}$$

Please note that in Theorem 4.6 the error term looks different: this is because our authors, in their finite wisdom, used a different value of h. This makes comparisons between methods a little dicey. In fact, midpoint is, as derived above, generally about twice as good as trapezoidal. Here is Theorem 4.6, nonetheless, in which they introduce points (indexed by -1, for example) that they will not use in the calculation. I've altered it slightly, to emphasize the $2h$ (because that's the fair comparison with trapezoidal):

Theorem 4.6: Let $f \in C^2[a,b]$, n be even, $h = \frac{b-a}{n+2}$, and $x_i = a+(i+1)h$ for $i = -1, \ldots, n+1$. Then $\exists \mu \in (a,b)$ for which

$$\int_a^b f(x)dx = 2h[0\ 1\ 1\ \cdots\ 1\ 0]\overline{f} + \frac{b-a}{24}(2h)^2 f''(\mu)$$

Composite Simpson's Rule: dividing the interval $[a,b]$ into n panels (but $2n+1$ points $x_i,\ i = 0, ..., n$), with stepsize $h = \frac{b-a}{n}$: on each we use elemental Simpson's, so that we get the following $n \times (2n+1)$ matrix that multiplies the vector of points $\overline{f} = (f(x_0), \ldots, f(x_n))$:

$$W = \frac{h}{6}\begin{bmatrix} 1 & 4 & 1 & & & & & & \\ & & 1 & 4 & 1 & & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & & 1 & 4 & 1 \end{bmatrix}_{n \times 2n+1}$$

The estimate is then given by

$$I \approx [1\ 1\ \cdots\ 1]_{1\times n} W_{n\times 2n+1} \overline{f}_{2n+1\times 1}$$

As usual, we can do that product once and for all:

$$I \approx \frac{h}{6}[1\ 4\ 2\ 4\ 2\ \cdots\ 2\ 4\ 2\ 4\ 1]\overline{f}$$

I call this the "1-4-2-4-2-step": just one of the classic dances that arise in numerical analysis.

Example: #12, p. 204

The derivation of the error terms is a wee bit more complicated: for Simpson's rule we make an "elemental error" of the form

$$-\frac{(h/2)^5 f^{(4)}(\xi_i)}{90}$$

on each panel (remember that I'm using the same step-size throughout: the text's value of h would is my $h/2$). Hence the total error for the interval is

$$E = -\sum_{i=1}^{n} \frac{h^5 f^{(4)}(\xi_i)}{90 \cdot 2^5}$$

which can be written as (see Theorem 4.4)

$$E = -\frac{(b-a)(h/2)^4 f^{(4)}(\mu)}{180}$$

or

$$E = -\frac{h^2(b-a)}{12}\left[\frac{h^2 f^{(4)}(\mu)}{240}\right]$$

So that we'd expect Simpson's to be about $\frac{h^2}{240}$ as good as trapezoidal, given that the respective derivatives are on the same order, and that we're comparing n panels to n panels.

Now, let's look at Simpson's error term (for example) in a slightly different way:

$$\sum_{i=1}^{n} h f^{(4)}(\xi_i)$$

is a Riemann sum for the integral

$$\int_a^b f^{(4)}(x)dx$$

so that

$$\int_a^b f^{(4)}(x)dx = f^{(3)}(b) - f^{(3)}(a)$$

so that

$$E \approx -\frac{(h/2)^4}{180}(f^{(3)}(b) - f^{(3)}(a))$$

Similar tricks works for the other elemental rules. This is nice, and gives us a guess for the size of the error; but it's not so helpful for bounding an error. For that, the original form is preferred.

Example: #20, p. 205

Example: #24, p. 205