MAT360 Section Summary: 4.6

Adaptive Quadrature

1. Summary

Objective: to approximate

$$I \equiv \int_a^b f(x)dx$$

to within $\epsilon > 0$.

The trick here is the usual one, of balancing errors, and trying for a better approximation by applying several methods and then taking an appropriate combination of them.

For example, we might try a Simpson's rule on a single panel over $[a, b]$ using a step-size of $h = (b - a)/2$:

$$I = I_h + E_h$$

where $I_h = S(a, b)$, and the error of the elemental Simpson's rule is

$$E_h = -\frac{h^5 f^{(4)}(\xi_h)}{90}$$

If we're lucky, this is already within ϵ of I. But what's the chance of that? And how would we know if we **are** within ϵ?

We need something to compare this estimate to, in order to get an idea of how well we're doing. Let's split our interval (a, b) in half, and compute the sum of **two** elemental Simpson's estimates with $h/2$ (that is, we consider a composite Simpson's – but it's just as easy to work with two elementals).

We now consider the error of the $h/2$ method:

$$I = I_{h/2} + E_{h/2}$$

where $I_{h/2} = S(a, a + h) + S(a + h, b)$ and

$$E_{h/2} = -\frac{(h/2)^5 f^{(4)}(\xi_L)}{90} - \frac{(h/2)^5 f^{(4)}(\xi_R)}{90} = -\frac{1}{16}\frac{h^5}{90}\left(\frac{f^{(4)}(\xi_L) + f^{(4)}(\xi_R)}{2}\right)$$

or, provided $f^{(4)}(x)$ is continuous on the inteval, we can invoke the IVT to conclude that

$$E_{h/2} = -\frac{1}{16}\frac{h^5 f^{(4)}(\xi_{h/2})}{90}$$

Now, provided $f^{(4)}(x)$ doesn't vary wildly on the interval, we can hope that $f^{(4)}(\xi_{h/2}) \approx f^{(4)}(\xi_h)$, so that the error of the $h/2$ composite method will be about a sixteenth of the error of the elemental rule:

$$E_{h/2} \approx \frac{E_h}{16}$$

Since the methods are both approximations to the same quantity, we can try to combine them to (approximately) eliminate their errors and so make a better approximation:

$$I \approx \frac{16I_{h/2} - I_h}{15}$$

On the other hand, we can also to check to see if the difference is significant enough to justify using the smaller step-size; if not, we can stick with the larger step-size – maybe even make it larger! That is, we can **adapt** to realities "on the ground" (or on the interval, at any rate!).

So

$$I - I_{h/2} = E_{h/2} \approx \frac{1}{16} E_h = \frac{1}{16}(I - I_h)$$

If we cavalierly assume that the $\approx$ in the above expression can be replaced by $=$, then substituting for I we find that

$$E_{h/2} = \frac{1}{16}(I - I_h) = \frac{1}{16}(I_{h/2} + E_{h/2} - I_h)$$

or

$$16E_{h/2} = I_{h/2} - I_h + E_{h/2}$$

Hence

$$E_{h/2} = \frac{I_{h/2} - I_h}{15}$$

We can easily measure the quantity on the RHS, and so determine if

$$E_{h/2} = \frac{|I_{h/2} - I_h|}{15} < \epsilon$$

If this condition is satisfied, then $I_{h/2}$ is sufficiently close: otherwise, we **divide and conquer**: split the error in half ($\epsilon/2$), and give one half of the error to each half of the $h/2$ method; then iterate until we've satisfied the error condition on each sub-sub-sub interval, or until exhausted.

If exhausted, we should really provide a warning that somewhere along the line our error condition wasn't met, and that consequently the original ϵ error may not have been met, either.

One last thing: for each interval on which $I_{h/2}$ is sufficiently close, there's no reason why you wouldn't go through the last little bit of effort to provide the following best estimate for the integral value:

$$I \approx \frac{16I_{h/2} - I_h}{15}$$

and so hope to get $O(h^6)$ error....