# MAT360 Section Summary: 5.2
Euler's Method

1. Summary

This first method for solving a well-posed initial value problem (IVP) is as simple as can be: we approximate a derivative with a finite difference. That's it!

Alternatively, Euler's method can be derived using the Taylor series expansion, and that is perhaps a better approach, since it can be generalized, and since it can be studied for the magnitude of the error we're going to be making.

2. **Definitions**

   - **mesh points**: the points (times) along the interval $[a, b]$ at which the solution will be approximated.

   - **step-size**: the interval of time between mesh points, denoted $h$:

$$t_i = a + ih \quad \text{for } i = 0, \ldots, N$$

   in which case $h = (b - a)/N$.

3. **Theorems/Formulas**

So, using Taylor, we have that

$$y(t_{i+1}) \equiv y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i)$$

and, since $y$ satisfies the differential equation,

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}y''(\xi_i)$$

We simply drop the error term, and hope that we don't make too bad a mistake, to generate the succession of iterates

$$\begin{aligned} &w_0 = \alpha \\ &w_{i+1} = w_i + hf(t_i, w_i) \quad \text{for } i = 0, \ldots, N-1 \end{aligned}$$

This is a **difference equation** associated with the given differential equation. Its solution, we hope, will be relatively close to the solution of the IVP. Hope aside, how bad can things get? What's the worst that can happen? The answer is in the following theorem:

**Theorem 5.9** (error bound): Suppose $f$ is continuous and satisfies a Lipschitz condition with constant $L$ on

$$D = \{(t,y)|a \leq t \leq b, -\infty < y < \infty\}$$

and that a constant $M$ exists with

$$|y''(t)| \leq M, \quad \forall t \in [a,b].$$

Let $y(t)$ denote the unique solution to the IVP

$$y' = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = \alpha,$$

and $\{w_i\}$ be the Euler approximations. Then, for each $i = 0, \ldots, N$,

$$|y(t_i) - w_i| \leq \frac{hM}{2L}\left[e^{L(ih)} - 1\right].$$

In fact, however, we have round-off errors, so that we solve a system like this:

$$\begin{aligned} &u_0 = \alpha + \delta_0 \\ &u_{i+1} = u_i + hf(t_i, u_i) + \delta_{i+1} \quad \text{for } i = 0, \ldots, N-1 \end{aligned}$$

**Theorem 5.10** (error bound, with round-off): Suppose $f$ and $y$ satisfy the conditions of Theorem 5.9, and let $u_i$ be the succession of iterates above. Then if $|\delta_i| < \delta$ for each $i = 0, \ldots, N$, then

$$|y(t_i) - u_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[e^{L(ih)} - 1\right] + |\delta_0| e^{L(ih)}$$

for each $i = 0, \ldots, N$.

4. Properties/Tricks/Hints/Etc.

One obvious strategy for improving our approximations is to make $h$ tremendously small. This may backfire, however, due to round-off error. The important result is contained in a result following Theorem 5.10: the minimal error of the error $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}$$

where $M$ is the bound on $y''(t)$ on $[a,b]$. How do we handle the second derivative, $y''(t)$? We don't know $y(t)$, how can we bound $y''(t)$?

We can use the chain rule: $y'(t) = f(t, y(t))$, so

$$y''(t) = \frac{d}{dt} f(t, y(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} y'(t)$$

Therefore

$$y''(t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f(t, y(t))$$

which we may be able to bound.