

# MAT360 Section Summary: 5.4

## Runge-Kutta Methods

### 1. Summary

I want you to see and understand the idea behind Runge-Kutta methods, which will be used in place of the Taylor methods. But the Taylor methods provide the impetus for these methods – they're a critical place to start.

The problem with the Taylor methods is that we have to compute all kinds of partial derivatives to make the methods work. Runge-Kutta methods will approximate those partial derivative terms by a succession of approximations computed using only the function  $f(t, y(t))$ .

For the Taylor method of order 2, we start with

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{3!}y'''(\xi_i)$$

in which we estimate  $y''(t_i)$  as

$$y''(t_i) = \frac{\partial f}{\partial t}(t_i, y(t_i)) + \frac{\partial f}{\partial y}(t_i, y(t_i))f(t_i, y(t_i))$$

This leads to the scheme

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + h\phi(t_i, w_i) \quad \text{for } i = 0, \dots, N \end{aligned}$$

for which

$$\phi(t_i, y_i) = f(t_i, y_i) + \frac{h}{2} \frac{\partial f}{\partial t}(t_i, y_i) + \frac{h}{2} f(t_i, y_i) \frac{\partial f}{\partial y}(t_i, y_i) \quad (1)$$

But those partial derivatives sure are a pain: any way we can get there more easily?

Yes, but we have to go through some more partial derivative pain: we consider Taylor's Theorem extended to functions of two variables:

**Theorem 5.13** Suppose that  $f(t, y)$  has continuous partial derivatives of order less than or equal to  $n + 1$  on  $D = \{(t, y) | a \leq t \leq b, c \leq y \leq d\}$ , and let  $(t_0, y_0) \in D$ . For every  $(t, y) \in D$ , there exists  $\xi$  between  $t_0$  and  $t$  and there exists  $\mu$  between  $y_0$  and  $y$  with

$$f(t, y) = P_n(t, y) + R_n(t, y)$$

where

$$\begin{aligned} P_n(t, y) &= f(t_0, y_0) + \\ & (t - t_0) \frac{\partial f}{\partial t}(t_0, y_0) + (y - y_0) \frac{\partial f}{\partial y}(t_0, y_0) + \\ & \frac{(t - t_0)^2}{2} \frac{\partial^2 f}{\partial t^2}(t_0, y_0) + (t - t_0)(y - y_0) \frac{\partial^2 f}{\partial t \partial y}(t_0, y_0) + \frac{(y - y_0)^2}{2} \frac{\partial^2 f}{\partial y^2}(t_0, y_0) + \dots + \\ & \frac{1}{n!} \sum_{j=0}^n \binom{n}{j} (t - t_0)^{n-j} (y - y_0)^j \frac{\partial^n f}{\partial t^{n-j} \partial y^j}(t_0, y_0) \end{aligned} \quad (2)$$

and

$$R_n(t, y) = \frac{1}{(n+1)!} \sum_{j=0}^{n+1} \binom{n+1}{j} (t-t_0)^{n+1-j} (y-y_0)^j \frac{\partial^{n+1} f}{\partial t^{n+1-j} \partial y^j}(\xi, \mu)$$

Now we're ready for the trick: this is the use of the Taylor series that you might not have thought of, out there on your desert island, even if you'd sat there for a few years. We want to replace  $\phi$  in (1) with a value of  $f$  computed at an intermediate location, rather than on the endpoint, as Euler did. That is, consider  $f(t + \alpha, y(t) + \beta)$ , expanded using the Taylor series:

$$f(t + \alpha, y(t) + \beta) = f(t, y(t)) + \alpha \frac{\partial f}{\partial t}(t, y(t)) + \beta \frac{\partial f}{\partial y}(t, y(t)) + R_1(t + \alpha, y(t) + \beta)$$

while

$$\phi(t, y(t)) = f(t, y(t)) + \frac{h}{2} \frac{\partial f}{\partial t}(t, y(t)) + \frac{h}{2} f(t, y(t)) \frac{\partial f}{\partial y}(t, y(t))$$

The expansion certainly resembles  $\phi$ , and if we make a good choice of  $\alpha$  and  $\beta$ , we can make them equal (up to the term  $R_1$ , which we're going to drop as our error):

$$\alpha = \frac{h}{2}$$

$$\beta = \frac{h}{2} f(t, y(t))$$

Given this choice,  $R_1$  is  $O(h^2)$ , just like Taylor 2; we call this method Runge-Kutta-2, (aka RK-2, or the Midpoint method):

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + hf(t_i + \frac{h}{2}, w_i + \frac{h}{2}f(t_i, w_i)) \quad \text{for } i = 0, \dots, N \end{aligned}$$

This sort of trick can be repeated for higher-order Taylor methods, although it's usually only continued up to RK-4:

$$\begin{aligned} w_0 &= \alpha \\ k_1 &= hf(t_i, w_i) \quad (\text{Euler - "RK-1"}) \\ k_2 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1) \quad (\text{RK-2}) \\ k_3 &= hf(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2) \\ k_4 &= hf(t_{i+1}, w_i + k_3) \\ w_{i+1} &= w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

You might recognize the sum  $\frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$  in the formula for  $w_{i+1}$  as the composite trapezoidal scheme for approximating the integral

$$\int_{t_i}^{t_{i+1}} y'(t) dt$$

This scheme is  $O(h^4)$ , just like Taylor-4, only we didn't need to compute a single partial derivative. It's wonderful! This method, therefore, will extend easily to systems of initial value problems.

We can revisit our pendulum example, for example, to see if we see any improvement in the solution comparing RK-4 to Euler. We'll also demonstrate how our ODE solves handle a system of DEs (the pendulum problem), as well as complexify the pendulum problem by adding damping (e.g. air resistance).