20 THE KOENIGSBERG BRIDGES

LEONHARD EULER • July 1953
Edited by JAMES R. NEWMAN

Leonhard Euler, the most eminent of Switzerland's scientists, was a gifted 18th-century mathematician who enriched mathematics in almost every department and whose energy was at least as remarkable as his genius. "Euler calculated without apparent effort, as men breathe, or as eagles sustain themselves in the wind," wrote François Arago, the French astronomer and physicist. It is said that Euler "dashed off memoirs in the half-hour between the first and second calls to dinner." According to the mathematical historian Eric Temple Bell he "would often compose his memoirs with a baby in his lap while the older children played all about him"—the number of Euler's children was 13. At the age of 28 he solved in three days a difficult astronomical problem which astronomers had agreed would take several months of labor; this prodigious feat so overtaxed his eyesight that he lost the sight of one eye and eventually became totally blind. But his handicap in no way diminished either the volume or the quality of his mathematical output. His writings will, it is estimated, fill 60 to 80 large quarto volumes when the edition of his collected works is completed.

The memoir published below is Euler's own account of one of his most famous achievements: his solution of the celebrated problem of the Koenigsberg bridges. The problem is a classic exercise in the branch of mathematics called topology (see "Topology," by Albert W. Tucker and Herbert S. Bailey, Jr., page 134 in this volume). Topology is the geometry of distortion; it deals with the properties of an object that survive stretching, twisting, bending or other changes of its size or shape. The Koenigsberg puzzle is a so-called network problem in topology.

In the town of Koenigsberg (where the philosopher Immanuel Kant was born) there were in the 18th century seven bridges which crossed the river Pregel. They connected two islands in the river with each other and with the opposite banks. The townsfolk had long amused themselves with this problem: Is it possible to cross the seven bridges in a continuous walk without recrossing any of them? When the puzzle came to Euler's attention, he recognized that an important scientific principle lay concealed in it. He applied himself to discovering this principle and shortly thereafter presented his simple and ingenious solution. He provided a mathematical demonstration, as some of the townsfolk had already proved to their own satisfaction by repeated trials, that the journey is impossible. He also found a rule which answered the question in general, whatever the number of bridges.

The Koenigsberg puzzle is related to the familiar exercise of trying to trace a given figure on paper without lifting the pencil or retracing a line. In graph form the Koenigsberg pattern is represented by the drawing on the left at the bottom of this page. Inspection shows that this pattern cannot be traced with a single stroke of the pencil. But if there are eight bridges, the pattern is the one at the right, and this one can be traced in a single stroke.

Euler's memoir gives a beautiful explanation of the principles involved and furnishes an admirable example of the deceptive simplicity of topology problems.—James R. Newman

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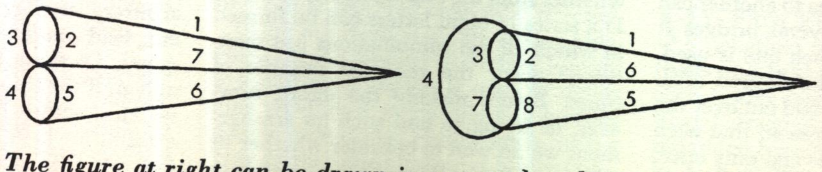
THE BRANCH of geometry that deals with magnitudes has been zealously studied throughout the past, but there is another branch that has been almost unknown up to now; Leibnitz spoke of it first, calling it the "geometry of position" (*geometria situs*). This branch of geometry deals with relations dependent on position alone, and investigates the properties of position; it does not take magnitudes into consideration, nor does it involve calculation with quantities. But as yet no satisfactory definition has been given of the problems that belong to this geometry of position or of the method to be used in solving them. Recently there was announced a problem which, while it certainly seemed to belong to geometry, was nevertheless so designed that it did not call for the determination of a magnitude, nor could it be solved by quantitative calculation; consequently I did not hesitate to assign it to the geometry of position, especially since the solution required only the consideration of position, calculation being of no use. In this paper I shall give an account of the method that I discovered for solving this type of problem, which may serve as an example of the geometry of position.

The problem, which I understand is quite well known, is stated as follows: In the town of Koenigsberg in Prussia there is an island A, called Kneiphof, with the two branches of the river Pregel flowing around it. There are seven bridges—*a*, *b*, *c*, *d*, *e*, *f* and *g*—crossing the two branches [*see illustration at the top of page 143*]. The question is whether a person can plan a walk in such a way that he will cross each



The figure at right can be drawn in one stroke; the one at left cannot

of these bridges once but not more than once. I was told that while some denied the possibility of doing this and others were in doubt, no one maintained that it was actually possible. On the basis of the above I formulated the following very general problem for myself: Given any configuration of the river and the branches into which it may divide, as well as any number of bridges, to determine whether or not it is possible to cross each bridge exactly once.

The particular problem of the seven bridges of Koenigsberg could be solved by carefully tabulating all possible paths, thereby ascertaining by inspection which of them, if any, met the requirement. This method of solution, however, is too tedious and too difficult because of the large number of possible combinations, and in other problems where many more bridges are involved it could not be used at all. . . . Hence I discarded it and searched for another more restricted in its scope; namely, a method which would show only whether a journey satisfying the prescribed condition could in the first instance be discovered; such an approach, I believed, would be simpler.

MY ENTIRE method rests on the appropriate and convenient way in which I denote the crossing of bridges, in that I use capital letters, A, B, C, D, to designate the various land areas that are separated from one another by the river. Thus when a person goes from area A to area B across bridge *a* or *b*, I denote this crossing by the letters AB, the first of which designates the area whence he came, the second the area where he arrives after crossing the bridge. If the traveler then crosses from B over bridge *f* into D, this crossing is denoted by the letters BD; the two crossings AB and BD performed in succession I denote simply by the three letters ABD, since the middle letter B designates the area into which the first crossing leads as well as the area out of which the second leads.

Similarly, if the traveler proceeds from D across bridge *g* into C, I designate the three successive crossings by the four letters ABDC. . . . The crossing of four bridges will be represented by five letters, and if the traveler crosses an arbitrary number of bridges his journey will be described by a number of letters which is one greater than the number of bridges. For example, eight letters are needed to denote the crossing of seven bridges.

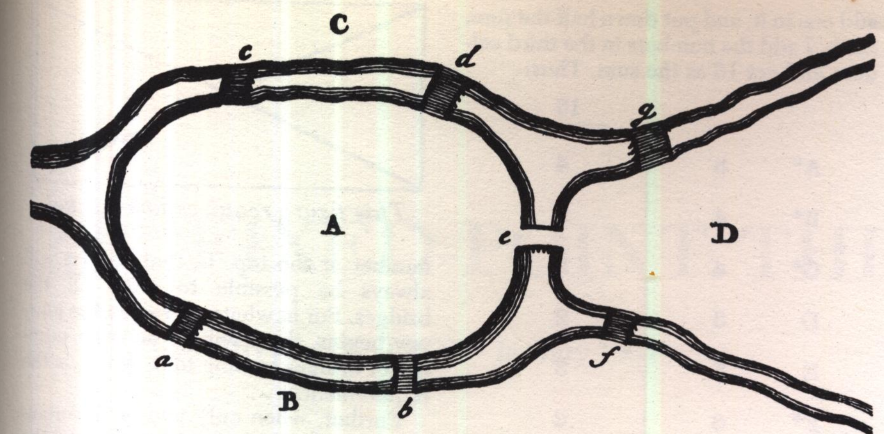
With this method I pay no attention to which bridges are used; that is to say, if the crossing from one area to another can be made by way of several bridges it makes no difference which one is used, so long as it leads to the desired area. Thus if a route could be laid out over the seven Koenigsberg bridges so that each bridge were crossed once and only once, we would be able to describe this route

Leonhard Euler (pronounced oiler); born Basel 1707; died Petrograd 1783

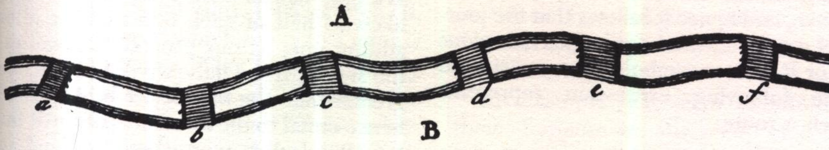
by using eight letters, and in this series of letters the combination AB (or BA) would have to occur twice, since there are two bridges, *a* and *b*, connecting the regions A and B. Similarly the combination AC would occur twice, while the combinations AB, BD, and CD would each occur once.

Our question is now reduced to whether from the four letters A, B, C and D a series of eight letters can be formed in which all the combinations just mentioned occur the required number of times. Before making the effort, however, of trying to find such an arrangement we do well to consider whether its existence is even theoretically possible or not. For if it could be shown that such an arrangement is in fact impossible, then the effort expended on finding it would be wasted. Therefore I have sought for a rule that would determine without difficulty, as regards this and all similar questions, whether the required arrangement of letters is feasible.

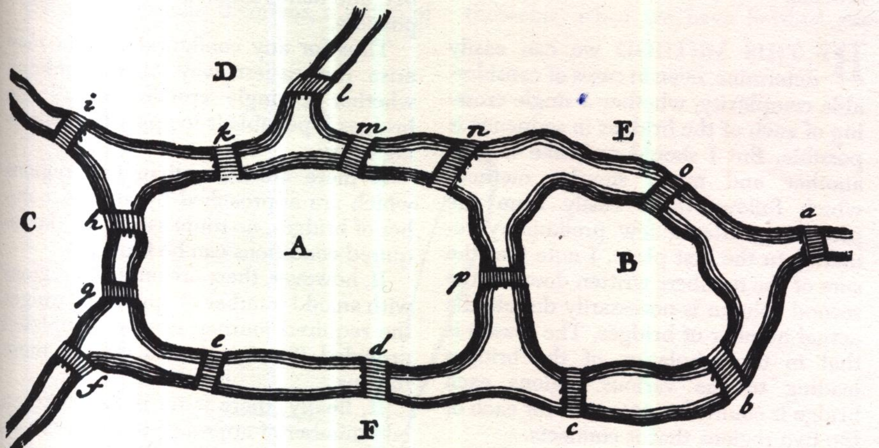
For the purpose of finding such a rule I take a single region A into which an arbitrary number of bridges, *a*, *b*, *c*, *d*, etc., lead [*middle illustration on the next page*]. Of these bridges I first consider only *a*. If the traveler crosses this bridge, he must either have been in A before crossing or have reached A after crossing, so that according to the above



Seven bridges of Koenigsberg crossed the River Pregel



Euler used a simpler case to elucidate his principle



This trip is possible though the Koenigsberg one is not

method of denotation the letter A will appear exactly once. If there are three bridges leading to A and the traveler crosses all three, then the letter A will occur twice in the expression for his journey, whether it begins at A or not. And if there are five bridges leading to A, the expression for a route that crosses them all will contain the letter A three times. If the number of bridges is odd, increase it by one, and take half the sum; the quotient represents the number of times the letter A appears.

LET US now return to the Koenigsberg problem [*top illustration above*]. Since there are five bridges leading to (and from) island A, the letter A must occur three times in the expression describing the route. The letter B must occur twice, since three bridges lead to B; similarly D and C must each occur twice. That is to say, the series of . . . letters that represents the crossing of the seven bridges must contain A three times and B, C and D each twice. But this is quite impossible with a series of eight letters [for the sum of the required letters is nine]. Thus it is apparent that a crossing of the seven bridges of Koenigsberg in the manner required cannot be effected.

Using this method we are always able, whenever the number of bridges leading to a particular region is odd, to determine whether it is possible in a journey to cross each bridge exactly once. Such a route exists if the number of bridges plus one is equal to the sum of the numbers which indicate how often each individual letter must occur. On the other hand, if this sum is greater than the number of bridges plus one, as it is in our example, then the desired route cannot be constructed. The rule that I gave for determining from the number of bridges that lead to A how often the letter A will occur in the route description is independent of whether these bridges all come from a single region B or from several regions, because I was considering only the region A, and attempting to determine how often the letter A must occur.

When the number of bridges leading to A is even, we must take into account whether the route begins in A or not. For example, if there are two bridges that lead to A and the route starts from A, then the letter A will occur twice—once to indicate the departure from A by one of the bridges and a second time to indicate the return to A by the other bridge. However, if the traveler starts his journey in another region, the letter A will occur only once, since by my method of description the single occurrence of A indicates an entrance into as well as a departure from A.

Suppose, as in our case, there are four bridges leading into the region A, and the route is to begin at A. The letter A will then occur three times in the expression for the whole route, while if the journey had started in another region, A would occur only twice. With six bridges leading to A, the letter A will occur four times if A is the starting point, otherwise only three times. In general, if the number of bridges is even, the number of occurrences of the letter A, when the starting region is not A, will be half the number of the bridges; when the route starts from A, one more than half.

Every route must, of course, start in some one region. Thus from the number of bridges that lead to each region I determine the number of times that the corresponding letter will occur in the expression for the entire route as follows: When the number of the bridges is odd, I increase it by one and divide by two; when the number is even, I simply divide it by two. Then if the sum of the resulting numbers is equal to the actual number of bridges plus one, the journey can be accomplished, though it must start in a region approached by an odd number of bridges. But if the sum is one less than the number of bridges plus one, the journey is feasible if its starting point is a region approached by an even number of bridges, for in that case the sum is again increased by one.

MY PROCEDURE for determining whether in any given system of rivers and bridges it is possible to cross each bridge exactly once is as follows: First I designate the individual regions separated from one another by the water as A, B, C, etc. Second, I take the total number of bridges, increase it by one, and write the resulting number at the top of the paper. Third, under this number I write the letters A, B, C, etc., in a column, and opposite each letter I note the number of bridges that lead to that particular region. Fourth, I place an asterisk next to each letter that has an even number opposite it. Fifth, in a third column I write opposite each even number the half of that number, and opposite each odd number I write half of the sum formed by that number plus one. Sixth, I add up the last column of numbers. If the sum is one less than, or equal to, the number written at the top, I conclude that the required journey can be made. But it must be noted that when the sum is one less than the number at the top, the route must start from a region marked with an asterisk, and . . . when these two numbers are equal, it must start from a region that does not have an asterisk.

For the Koenigsberg problem I would set up the tabulation as follows:

Number of bridges 7,
giving 8 (=7+1)

A	5	3
B	3	2
C	3	2
D	3	2

The last column now adds up to more than 8, and hence the required journey cannot be made.

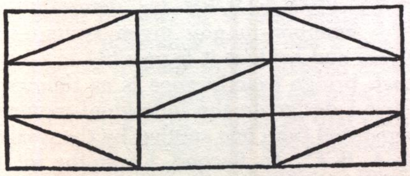
Let us take an example of two islands with four rivers forming the surrounding water [*bottom illustration on the preceding page*]. Fifteen bridges, marked *a*, *b*, *c*, *d*, etc., across the water around the islands and the adjoining rivers. The question is whether a journey can be arranged that will pass over all the bridges, but not over any of them more than once. I begin by marking the regions that are separated from one another by water with the letters A, B, C, D, E, F—there are six of them. Second, I take the number of bridges (15) add one and write this number (16) uppermost. Third, I write the letters A, B, C, etc., in a column and opposite each letter I write the number of bridges connecting with that region, *e.g.*, eight bridges for A, four for B, etc. Fourth, the letters that have even numbers opposite them I mark with an asterisk. Fifth, in a third column I write the half of each corresponding even number, or, if the number is odd, I add one to it, and put down half the sum. Sixth, I add the numbers in the third column and get 16 as the sum. Thus:

		16
A*	8	4
B*	4	2
C*	4	2
D	3	2
E	5	3
F*	6	3
		16

The sum of the third column is the same as the number 16 that appears above, and hence it follows that the journey can be effected if it begins in regions D or E, whose symbols have no asterisk. The following expression represents such a route:

EaFbBcFdAeFfCgAhCiDkAmEnAp-
BoElD.

Here I have indicated, by small letters between the capitals, which bridges are crossed.

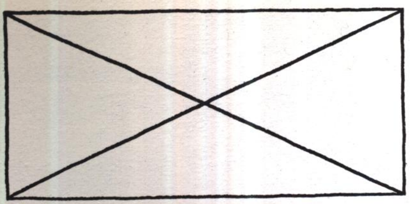


This figure requires only one stroke

BY THIS METHOD we can easily determine, even in cases of considerable complexity, whether a single crossing of each of the bridges in sequence is possible. But I should now like to give another and much simpler method, which follows quite easily from the preceding, after a few preliminary remarks. In the first place, I note that the sum of the numbers written down in the second column is necessarily double the actual number of bridges. The reason is that in the tabulation of the bridges leading to the various regions each bridge is counted twice, once for each of the two regions that it connects.

From this observation it follows that the sum of the numbers in the second column must be an even number, since half of it represents the actual number of bridges. Hence . . . if any of the numbers opposite the letters A, B, C, etc., are odd, an even number of them must be odd. In the Koenigsberg problem for instance, all four of the numbers opposite the letters A, B, C, D, were odd, while in the example just given only two of the numbers were odd, namely those opposite D and E.

Since the sum of the numbers opposite A, B, C, etc., is double the number of bridges, it is clear that if this sum is increased by two in the latter example and then divided by two, the result will be the number written at the top. When all the numbers in the second column are even, and the half of each is written down in the third column, the total of this column will be one less than the number at the top. In that case it will always be possible to cross all the bridges. For in whatever region the journey begins, there will be an even number of bridges leading to it, which is the requirement. . . .



This figure requires two strokes

Further, when only two of the numbers opposite the letters are odd, and the others even, the required route is possible provided it begins in a region approached by an odd number of bridges. We take half of each even number, and likewise half of each odd number after adding one, as our procedure requires; the sum of these halves will then be one greater than the number of bridges, and hence equal to the number written at the top. But [when more than two, and an even number] of the numbers in the second column are odd, it is evident that the sum of the numbers in the third column will be greater than the top number, and hence the desired journey is impossible.

Thus for any configuration that may arise the easiest way of determining whether a single crossing of all the bridges is possible is to apply the following rules:

If there are more than two regions which are approached by an odd number of bridges, no route satisfying the required conditions can be found.

If, however, there are only two regions with an odd number of approach bridges the required journey can be completed provided it originates in one of these regions.

If, finally, there is no region with an odd number of approach bridges, the required journey can be effected, no matter where it begins.

These rules solve completely the problem initially proposed.

AFTER we have determined that a route actually exists we are left with the question how to find it. To this end the following rule will serve: Wherever possible we mentally eliminate any two bridges that connect the same two regions; this usually reduces the number of bridges considerably. Then—and this should not be difficult—we proceed to trace the required route across the remaining bridges. The pattern of this route, once we have found it, will not be substantially affected by the restoration of the bridges which were first eliminated from consideration—as a little thought will show. Therefore I do not think I need say more about finding the routes themselves.