1 Practice 7

Prove that, for any natural number n, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Proof: First we check $P(1)$:

$$1 = \frac{1(1+1)}{2}$$

Okay! Now the induction: suppose that $P(k)$ is true, and consider the case of $k + 1$:

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$$

but

$$\frac{k(k+1)}{2} + (k+1) = (k+1)(\frac{k}{2} + 1) = \frac{(k+1)(k+2)}{2}$$

Now

$$\frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}$$

so $P(k+1)$ is also true. Q.E.D., by mathematical induction.

2 Exercise 30/34, p. 108/106

Prove: that $2^{n-1} \leq n!$ for $n \geq 1$.

Proof:

1. Anchor: consider $n = 1$: $2^{1-1} = 2^0 = 1 \leq 1!$. Check.

2. Implication: suppose $P(k)$ true (that is, $2^{k-1} \leq k!$), $k \geq 1$, and consider $2^{(k+1)-1}$:

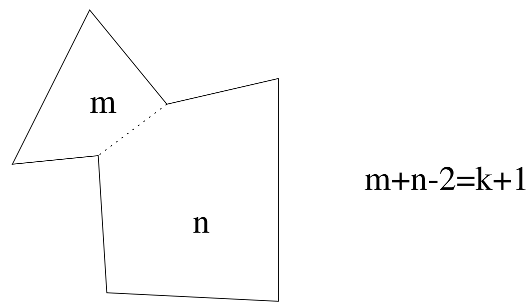
   $$2^{(k+1)-1} = 2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k! \leq (k+1) \cdot k! = (k+1)!$$

   Hence $P(k+1)$, and the result is proven by mathematical induction. Q.E.D.

3 Exercise 60/64b, p. 110/109

Prove that the sum of the interior angles of an n-sided simple closed polygon is $180(n-2)$ for all $n \geq 3$.

First of all, we check the base case: since the sum of the interior angles of a triangle is 180, we have $P(3)$. Suppose that $P(r)$ is true for all $r \leq k$, and consider the case of a $k+1$-sided polygon ($k \geq 3$): the idea is to "divide and conquer". We split the $k+1$-sided polygon into two polygons by connecting



two of the unconnected vertices in such way that we create two new polygons, of m and n sides each. Now $m+n-2 = k+1$, since we double count the new side in the two smaller polygons. Since $m \leq k$ and $n \leq k$, they satisfy our hypothesis, and we can write the sum of their interior angles as

$$180(m-2) + 180(n-2) = 180(m+n-4) = 180((k+1)-2)$$

The sum of the interior angles of the two sub-polygons is equal to the sum of the interior angles of the initial polygon, since we have simply subdivided two of the angles in the initial polygon and counted part of the measure in one of the new sub-polygons and the rest in the other. Hence, P(k+1); Q.E.D., by mathematical induction.

4 Exercise 68, p. 186 (not in Edition 5)

Prove: that the first principle implies well-ordering.

Proof: by contradiction. The *principle of well-ordering* says that every non-empty set of positive integers contains a smallest member. Given a non-empty set of positive integers T without a smallest member, we will prove that this is a contradiction if the first principle of induction holds.

Since T is non-empty, it contains some positive integer: call it a. Consider $P(n)$ defined by "every member of T is greater than n". Clearly $P(1)$, since 1 would be the smallest member otherwise. Suppose $P(k)$ to be true, and consider $P(k+1)$: since all members of T are greater than k, if $k+1$ is in T, then $k+1$ is the smallest member. Since there is no smallest member, $k+1$ is not in T, and hence $P(k+1)$. Thus, by the first principle of mathematical induction, we have demonstrated that $P(n)$ is true for all $n \geq 1$.

But $P(a)$ is false, since a is in the set! This is a contradiction, which means that T did, in fact, have a smallest member, which establishes the principle of well-ordering.

5 Backwards induction (The prisoner's last request)

A prisoner, condemned to die by the Sultan of an antique land, asked for one last request: "Please Sultan, if you would only grant me two favors: one, that you have me executed in the month of January (next month), and two, that you don't allow me to know the day of my death until 10 a.m. of the day I am to die." The Sultan, being a merciful man, granted these requests, whereupon the prisoner demonstrated that it was impossible to execute him subject to these conditions. How?