A "Bouquet" of Discontinuous Functions for Beginners in Mathematical Analysis

Giacomo Drago, Pier Domenico Lamberti, and Paolo Toni

Abstract. We present a selection of a few discontinuous functions and we discuss some pedagogical advantages of using such functions in order to illustrate some basic concepts of mathematical analysis to beginners.

1. INTRODUCTION. In this paper we present a selection of several discontinuous real-valued functions of one real variable which we believe could be proposed to any beginner in mathematical analysis, even to students of secondary school dealing with the first notions of calculus. Some of them are elementary and well known, others a bit more sophisticated. Most of them are modeled on the Dirichlet function.

Our aim is to point out some pedagogical advantages of using discontinuous functions rather than classical analytic functions. Usually, young students tend to think of basic mathematical analysis as a set of rules of calculus and, surely, many undergraduate students would claim to feel more acquainted with the notion of derivative rather than with the notion of function. This is not surprising at all, if we consider that the historical evolution of the notion of function has been very long and troubled; see Youschkevitch [**15**] and Kleiner [**8, 9**]. See also Nicholas [**11**], Deal [**3**], and Thurston [**13**] for an interesting discussion about some pedagogical issues related to the definition of function. A modern definition of function can be found, e.g., in the classic book of Bourbaki [**2**], published more than two hundred years after the definition of Johann Bernoulli (1718). As is pointed out in [**15**, p. 79], a significant step in this process was the formulation of A. Cournot (1841), which we report here for the convenience of the reader:

> We understand that a quantity may depend on another [quantity], even in case the nature of this dependence is such that it cannot be expressed in terms of a combination of algebraic symbols.

This level of generality is commonly attributed to Dirichlet, who in 1829 proposed his celebrated function $\mathcal{D}$ defined on $(0, 1)$ as follows:

$$D(x) = \begin{cases} 1, & \text{if } x \in (0, 1) \cap \mathbb{Q}, \\ 0, & \text{if } x \in (0, 1) \setminus \mathbb{Q}. \end{cases} \qquad (1)$$

In fact, this example opened a door to a new world: functions are not just formulas, or analytic expressions, as was commonly assumed in the 18th century. Functions can be defined by very general laws.

From a pedagogical point of view, deciding the level of generality of functions to use with young students is not straightforward. According to Kleiner [**9**, pp. 187–188], it is possible to teach an elementary model of analysis by placing emphasis solely on curves and the equations that represent them, without necessarily talking about functions. Kleiner argues that students would find curves more natural than functions and

teachers should introduce the notion of function only when there is an evident need for it. Teaching mathematics should follow the historical evolution of mathematics itself: new definitions and concepts were introduced when the need arose, and it should be so in teaching too. Thus, the definition of function should be given gradually, following its historical path. At first, one should introduce functions as formulas, then as rules, and only at last, if required, as a set of ordered pairs (after all, as Kleiner [9] points out, "giving this latter definition and proceeding to discuss only linear and quadratic functions makes little pedagogical sense").
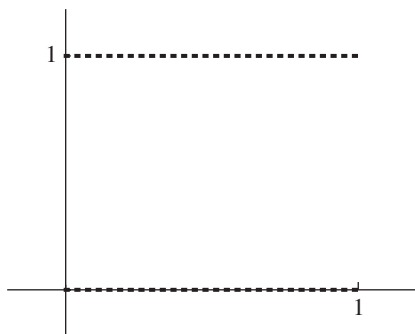
Figure 1. Graph of the Dirichlet function.

In [14] another approach has been developed. The introduction of [14] reads: "after all, analysis has few bases: the concepts of function, infinity, limit, continuity, and differentiability. To visit these locations in a comprehensive way, one should allow these concepts the faculty of expressing all of their potential and all their fantastic and incredible situations. To achieve this high level it is necessary to stick closely to the definition of function as a correspondence of free sets." The modern definition of function clearly allows a deeper comprehension of other concepts, such as those of limit, continuity, and differentiability. It is important that students understand the far-reaching features of these notions, because only in this way will they be able to avoid mistakes due to a limited concept of function. Students should be aware from the very beginning that formulas provide a very small class of functions: analyticity is a property enjoyed by very special functions. Being discontinuous is not synonymous with being pathological because most functions are discontinuous. In the same way, being continuous is not synonymous with being smooth since continuous functions can be very rough, as in the case of the celebrated Weierstrass functions.

Following this plan, in [14] the authors propose a collection of problems the solutions of which require the modern definition of function: in some cases, using highly discontinuous functions is not essential but it simplifies the argument and makes the situation clear.

We note that there is a vast literature devoted to so-called pathological functions. We mention the classic book by Gelbaum and Olmsted [6] and the recent extensive monograph by Kharazishvili [7]. We also mention Thim [12], which is basically a treatise concerning continuous nowhere differentiable functions. However, at a pedagogical level, not much material seems to be available.

In this paper, we adopt the approach of [14] and we discuss some peculiar examples, some of which are taken from [14]. The focus is on special discontinuous functions, the definitions of which are algebraically and technically simple in most cases. We believe that this approach could be used not only with good students but also with weak

students who may take advantage of examples which illustrate profound concepts but remain at a low level of complexity.

2. EXAMPLES OF DIRICHLET-TYPE FUNCTIONS.
In this section we present several discontinuous functions modeled on the Dirichlet function (1). We think that students may find such examples easy and eventually amusing, once they are acquainted with function (1). On the other hand, asking a student of a first-year calculus course to provide such examples could be very challenging.

1. A function continuous at only one point. If we ask a beginner in mathematical analysis whether a function from $\mathbb{R}$ to itself can be continuous at only one point, most of the time we will get the answer "no" as a consequence of an over-simplified view of the concept of continuity. Usually, students have an idea of continuity as a global property of a function: according to this point of view, continuous functions are those functions whose graphs can be drawn without lifting the chalk from the blackboard. This vision is not completely wrong since a function is continuous in the whole of an interval if and only if its graph is a pathwise-connected set in the plane. However, the following example clarifies the local nature of continuity.
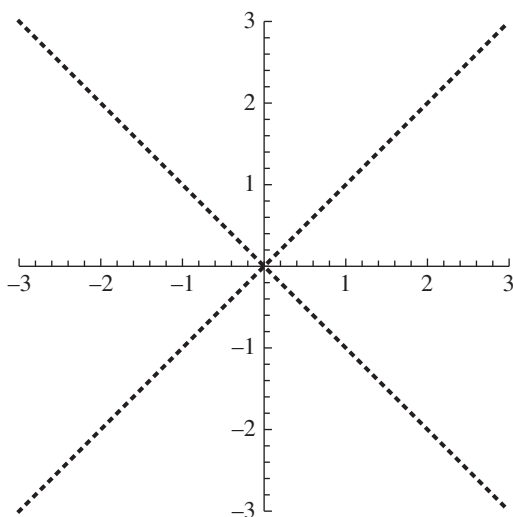

Figure 2. Graph of function F_1.

Let F_1 be the function from $\mathbb{R}$ to itself defined by

$$F_1(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ -x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

The function F_1 is continuous only at $x = 0$.

As a variant of the function F_1, one can consider the function G_1 from $\mathbb{R}$ to itself defined by

$$G_1(x) = \begin{cases} \sin x, & \text{if } x \in \mathbb{Q}, \\ -\sin x, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
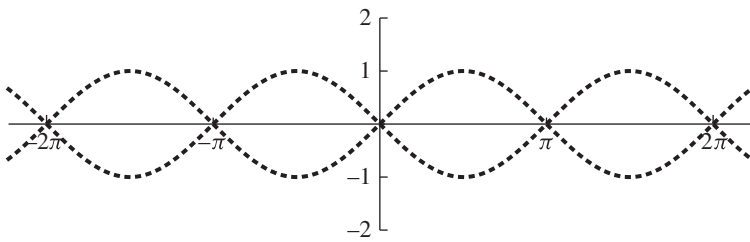
Figure 3. Graph of function G_1.

The function G_1 is continuous only at a countable set of points, precisely at the points $x = k\pi$ with $k \in \mathbb{Z}$.

2. A function differentiable at only one point. Loosely speaking, the derivative of a function at a point is the slope of the tangent line to the graph of the function at that point. However, the notion of differentiability has a local nature which goes much beyond the geometric idea of a tangent line.

Let F_2 be the function from $\mathbb{R}$ to itself defined by

$$F_2(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ -x^2, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

The function F_2 is differentiable only at $x = 0$ and is discontinuous at any point $x \neq 0$. Proving that F_2 is differentiable at $x = 0$ is an easy exercise. However, it may be instructive to deduce such a proof from the following elementary lemma.

Lemma 2. *Let f, g, and h be functions from $\mathbb{R}$ to itself such that $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be fixed. If f and g are differentiable at x_0, $f'(x_0) = g'(x_0)$, and $f(x_0) = g(x_0)$, then h is differentiable at x_0 and $h'(x_0) = f'(x_0) = g'(x_0)$.*
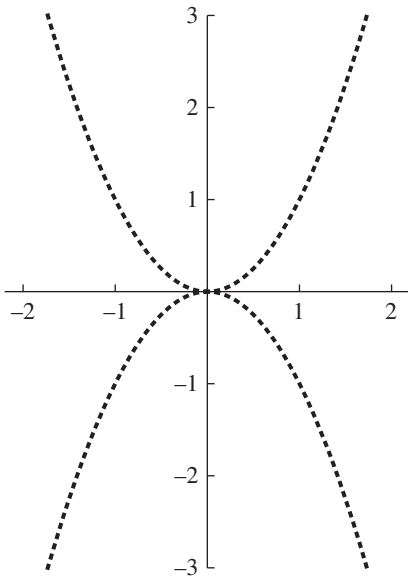

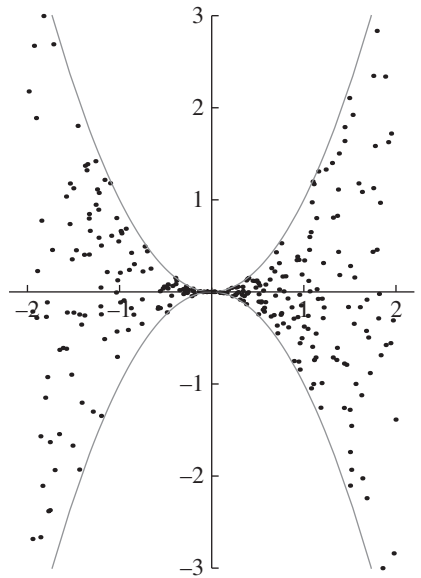**Figure 4.** Graph of function F_2.

Figure 5. A randomly generated function differentiable at $x = 0$.

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Students may find it entertaining to apply the previous lemma to randomly generated functions h satisfying the condition $-x^2 \le h(x) \le x^2$ for all x in a neighborhood of zero and find out that such functions are differentiable at zero (see Figure 5).

3. A function with positive derivative at one point, which is not monotone in any neighborhood of that point.

Let F_3 be the function from $\mathbb{R}$ to itself defined by

$$F_3(x) = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}, \\ 2x - 1, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

By Lemma 2, the function F_3 is differentiable at the point $x = 1$ and $F_3'(1) = 2$, but F_3 is not monotone in any neighborhood of 1. Those students who are used to identifying increasing functions with functions with positive derivative may find this example rather bizarre. However, this example points out a well-known subtle distinction concerning increasing functions. In fact, if a function f from $\mathbb{R}$ to itself is differentiable at a point x_0 and $f'(x_0) > 0$ then f is increasing at the point x_0 in the sense that for all x in a convenient neighborhood of x_0 we have: $f(x) > f(x_0)$ if $x > x_0$ and $f(x) < f(x_0)$ if $x < x_0$. Monotonicity is a stronger notion and, in the case of differentiable functions, it occurs when the derivative does not change sign in the whole of an interval.
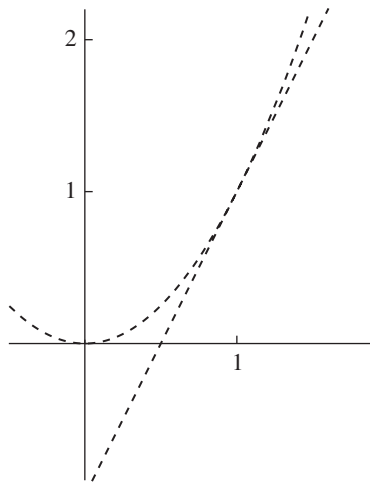

Figure 6. Graph of function F_3.

Variants of the function F_3 can be easily defined by considering convex combinations of x^2 and $2x - 1$ of the form

$$\theta(x)x^2 + (1 - \theta(x))(2x - 1),$$

where θ is a suitable function taking values in $[0, 1]$. A "continuous version" of F_3 is given for example by the function G_3 from $\mathbb{R}$ to itself defined by[1]

$$G_3(x) = \sin^2\left(\frac{9}{x-1}\right) \cdot x^2 + \left(1 - \sin^2\left(\frac{9}{x-1}\right)\right) \cdot (2x - 1),$$

[1] The choice of the number 9 in the definition of G_3 is not essential, and is only aimed at emphasizing the oscillations of the graph in Figure 7.

for all $x \in \mathbb{R} \setminus \{1\}$, and $G_3(1) = 1$. Again, $G_3'(1) = 2 > 0$, but G_3 is not monotone in any neighborhood of the point $x = 1$.

Clearly the function G_3 is much smoother than F_3; however, defining F_3 and proving its nonmonotonicity is simpler.
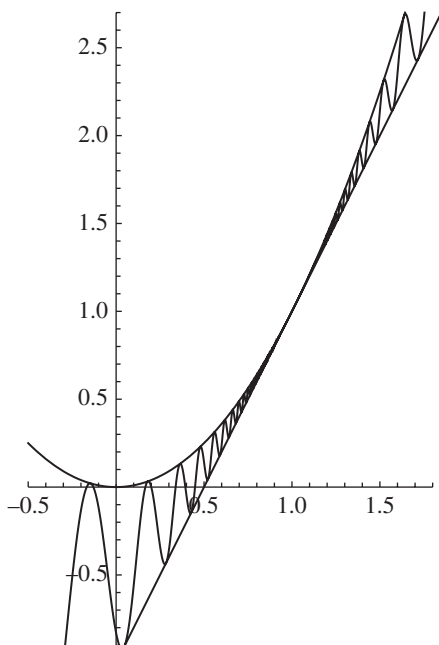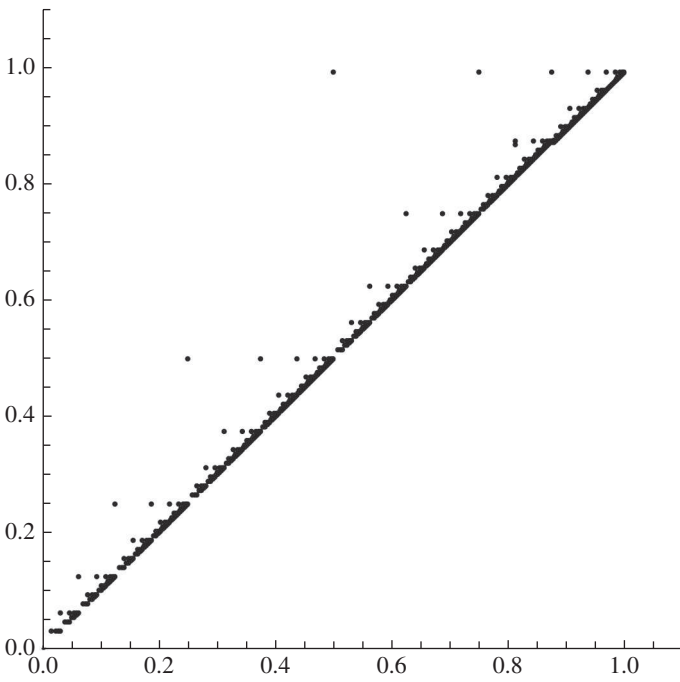

Figure 7. Graph of function G_3.

3. FURTHER EXAMPLES. In this section we present examples of functions with a specific type of discontinuity at any point of the domain, such as removable discontinuities or jump discontinuities. Although complicated, the definitions of these functions are constructive and make use of elementary notions. We note that imposing a specific type of discontinuity at any point of the domain of a function leads to limitations on the structure of the domain itself. In fact, the following result in Klippert [**10**] holds.

Theorem 3. *Let f be a function from $\mathbb{R}$ to itself. Let A be the set of those points $x_0 \in \mathbb{R}$ such that f is discontinuous at x_0 and at least one of the two limits $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exists and is finite. Then A is at most countable.*

4. A function with removable discontinuities at every point of its domain. The following function F_4 is defined on the dyadic rational numbers in $(0, 1)$, i.e., those numbers in $(0, 1)$ with a finite binary expansion. A simple formal definition is given below in (6). However, we prefer to begin by describing the elementary geometric construction given in [**14**], which could also be presented to beginners.

We construct the function F_4 by means of the following iterative procedure (see Figure 9).

Step 1. Let $V = 0$ and $L = 1$, and consider the square in the Euclidean plane with edges parallel to the coordinate axes whose lower left vertex is (V, V) and

Figure 8. Graph of function F_4 (arrested at the fourth iteration).

whose edges have length L. Let

$$B_{V,L} = \left\{ V + \sum_{j=1}^{n} \frac{L}{2^j} : n \in \mathbb{N} \right\}.$$

Then we set $F_4(x) = V + L$ for all $x \in B_{V,L}$.

Step 2. Inside the square defined in Step 1, consider the sequence of squares with edges of length $L_n = \frac{L}{2^n}$ and parallel to the coordinate axes, and lower left vertices with coordinates (V_n, V_n), where $V_n = V + \sum_{j=1}^{n-1} \frac{L}{2^j}$ for all $n \in \mathbb{N}$ with $n \geq 2$, and $(V_1, V_1) = (V, V)$.

Repeat Step 1 and Step 2, using each square obtained in Step 2 in place of the original square, and iterate the process.

The domain of the function F_4 defined by this procedure is $B = \cup_{V,L} B_{V,L}$, which is clearly the set of the dyadic rational numbers in $(0, 1)$.

Looking closely at the construction described above and at the few first iterations (see Figure 9), it is evident that

$$\lim_{x \to v} F_4(x) = v < F_4(v) \tag{4}$$

for all $v \in B$. Thus F_4 has a removable discontinuity at each point $v \in B$. Indeed, in order to remove the discontinuity at a point v, it is enough to redefine the function by setting $\tilde{F}_4(v) = v$.

It is possible to give a non-iterative definition of the function F_4. Indeed the set B can be represented as
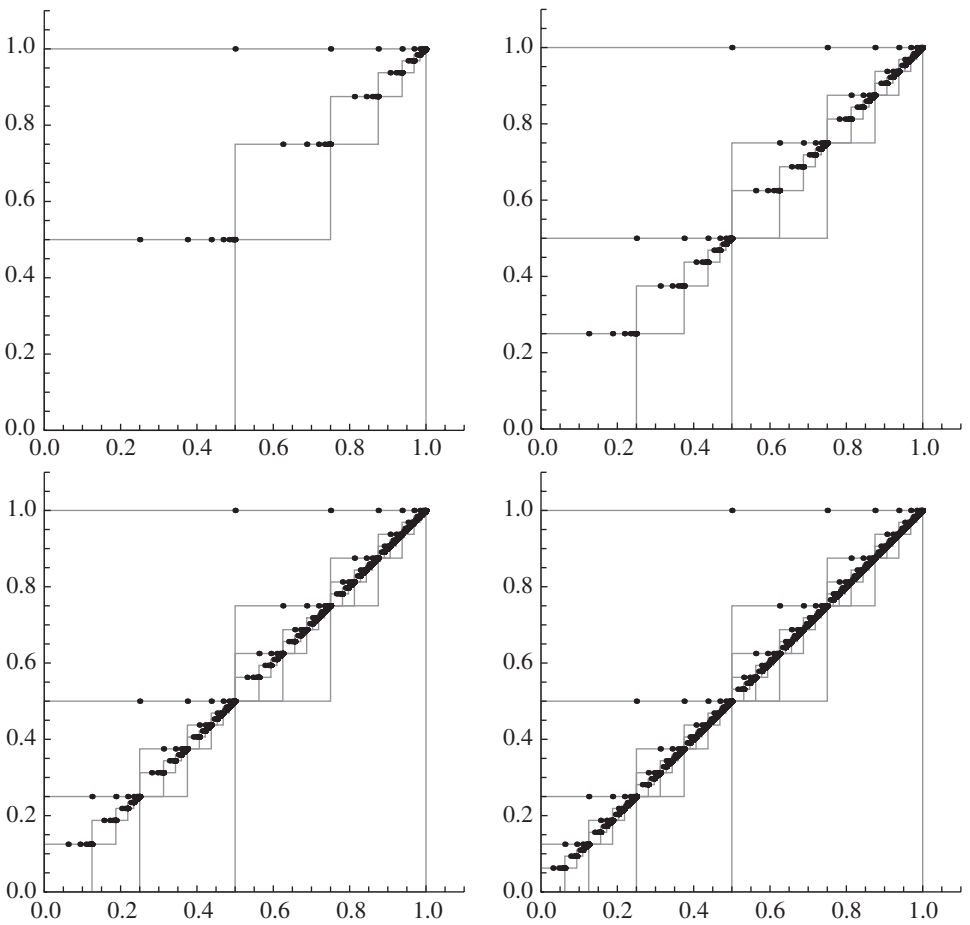
Figure 9. The first four iterations.

$$B = \left\{ \frac{p}{2^n} : n, p \in \mathbb{N}, \ p \text{ is odd}, \ p < 2^n \right\} \tag{5}$$

and F_4 can be defined on B directly by the equality

$$F_4 \left(\frac{p}{2^n} \right) = \frac{p+1}{2^n} \tag{6}$$

for all $n, p \in \mathbb{N}$, where p is odd and $p < 2^n$. This alternative definition allows an easy proof of (4) since the summand $1/2^n$ in the right-hand side of (6) vanishes in the limiting procedure (see also (8) and (9)).

A well-known function which enjoys a similar property can be found in several textbooks. It is defined as follows. Let G_4 be the function from $(0, 1) \cap \mathbb{Q}$ to itself defined by
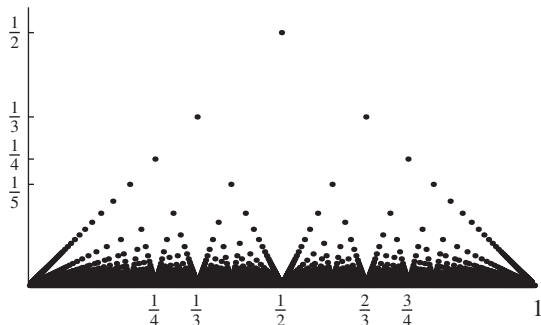
$$G_4 \left(\frac{p}{q} \right) = \frac{1}{q} \tag{7}$$

Figure 10. Graph of function G_4.

for all $p, q \in \mathbb{N}$, $0 < p/q < 1$, p, q coprime. It is not difficult to prove that

$$\lim_{x \to x_0} G_4(x) = 0 \neq G_4(x_0) \tag{8}$$

for all $x_0 \in (0, 1) \cap \mathbb{Q}$. The function G_4 can be extended to the whole of $(0, 1)$ by setting $G_4(x) = 0$ for all $x \in (0, 1) \setminus \mathbb{Q}$ and turns out to be continuous at any point $x \in (0, 1) \setminus \mathbb{Q}$. According to Appell [**1**, p. 10] some students would call function G_4 "Dirichlet light" because its set of discontinuities is "lighter" than the set of discontinuities of the Dirichlet function.

We note that by (6) and (7) it immediately follows that

$$F_4(x) = x + G_4(x) \tag{9}$$

for all $x \in B$. Equality (9) could be used to define F_4 via G_4 and to give a further proof of (4). In this way, the function F_4, whose original definition is based on a geometric construction, can eventually be represented via a function whose definition has a completely different origin.

5. A function with jump discontinuities at all points of its domain. Monotone functions with many jump discontinuities are well known in real analysis and probability theory: these functions can be obtained as cumulative distribution functions associated with discrete measures (see, e.g., Folland [**5**, p. 102]). Here we present two functions defined on the set B of the dyadic rational numbers in $(0, 1)$ with jump discontinuities at every point of B. These functions are defined directly by using binary notation and do not require any advanced tool.

We recall that the set B in (5) can be represented also in the form

$$B = \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \frac{x_j}{2^j} \;\middle|\; x_j \in \{0, 1\} \text{ for all } j \in \{1, \ldots, n\}, \; x_n = 1 \right\}. \tag{10}$$

Thus, given $x \in B$ there exists $n(x) \in \mathbb{N}$ such that $x = \sum_{j=1}^{n(x)} x_j/2^j$ for suitable values of $x_j \in \{0, 1\}$, where $x_{n(x)} = 1$; using binary notation x can be written as

$$0. \, x_1 \, x_2 \, \cdots \, x_{n(x)}.$$

Let F_5 be the function from B to itself defined by

$$F_5(0. \, x_1 \, x_2 \, \cdots \, x_{n(x)}) = 0. \, y_1 \, y_2 \, \cdots \, y_{m(x)},$$

where the digits $y_1, \ldots, y_{m(x)}$ are obtained from the digits $x_1, \ldots, x_{n(x)}$ of x by simply replacing each 0 with 01. For example, $F_5(0.01) = 0.011$, $F_5(0.101001) = 0.101101011$, $F_5(0.000011) = 0.0101010111$, and $F_5(0.1) = 0.1$. It is easy to prove that F_5 is a monotone function strictly increasing on B. Thus, F_5 admits left and right limits at any point of B. Moreover, it is not difficult to realize that in fact

$$F_5(v) = \lim_{x \to v^-} F_5(x) < \lim_{x \to v^+} F_5(x)$$

for all $v \in B$. Hence F_5 is left-continuous with a jump discontinuity at every point $v \in B$. In order to get rid of the monotonicity one can consider the function G_5 defined by

$$G_5 = F_4 \circ F_5.$$

The function G_5 is particularly interesting since it admits a geometric description in the spirit of the iterative construction of F_4: in this case one has to consider rectangles with no common vertices rather than squares (see Figures 11 and 12). The function G_5 is discussed in detail in Drago [**4**], where it is also proved by using binary representations that for all $v \in B$ the one-sided limits of G_5 at v exist and satisfy

$$\lim_{x \to v^-} G_5(x) < \lim_{x \to v^+} G_5(x) < G_5(v).$$
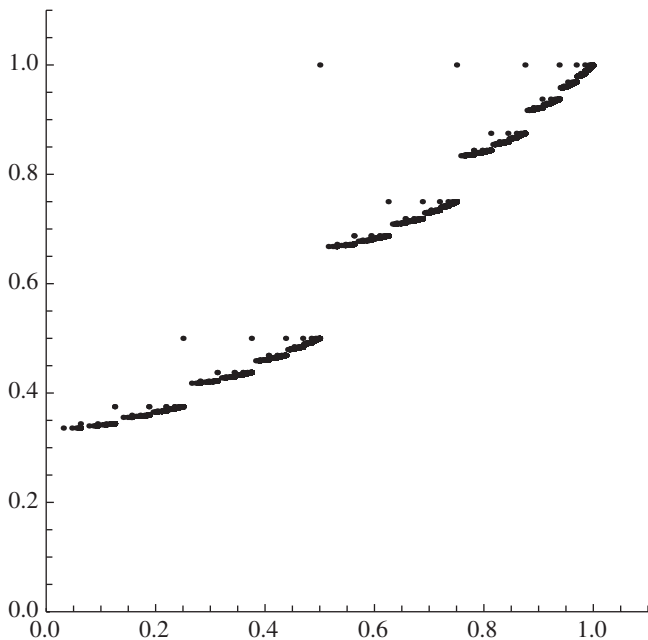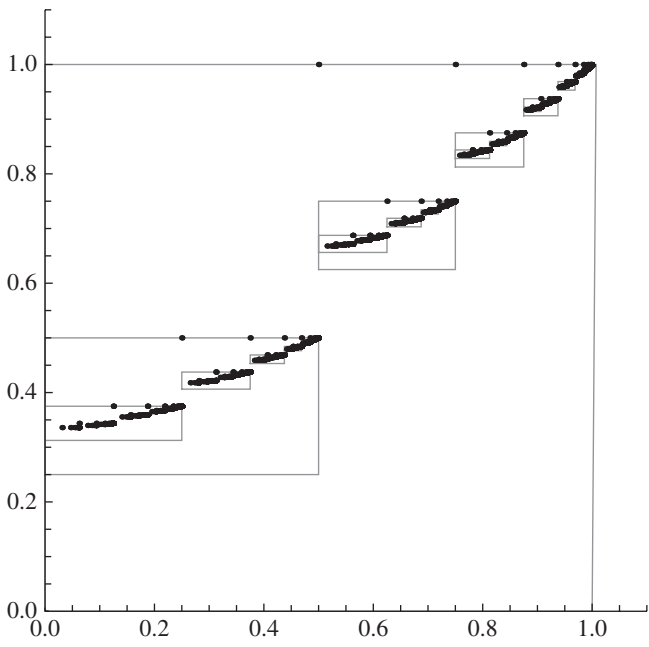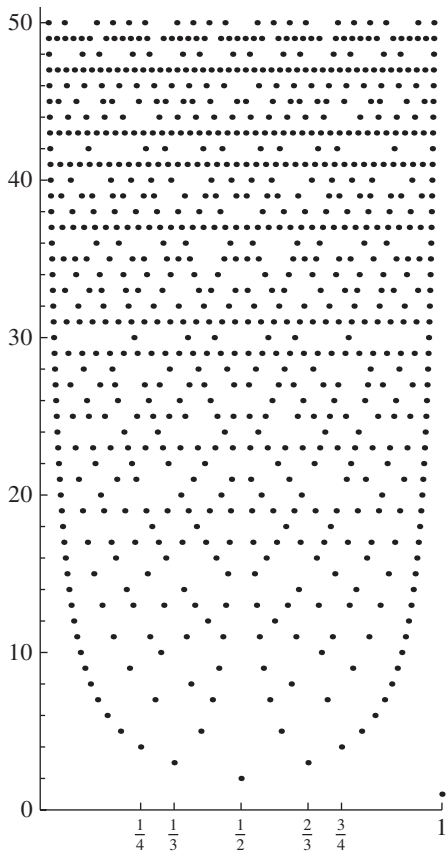

Figure 11. Graph of function G_5 (arrested at the third iteration).

6. A function approaching infinity at any point of its domain.

The following example is a simple variant of the function G_4. Let F_6 be the function from $(0, 1) \cap \mathbb{Q}$ to itself defined by

$$F_6\left(\frac{p}{q}\right) = q$$

Figure 12. Construction of function G_5.

Figure 13. Graph of function F_6.

for all $p, q \in \mathbb{N}, 0 < p/q < 1$, p, q coprime. It is not difficult to prove that

$$\lim_{x \to x_0} F_6(x) = +\infty$$

for all $x_0 \in [0, 1]$.

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A Ramanujan-Type Formula for $\left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}}/e$

Motivated by Sanjay Khattri's proofs [1] that $e < \left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}}$, I offer the following formula, which I think Ramanujan would have liked, for the ratio of the two sides of Khattri's inequality:

$$\left(1 + \frac{1}{n}\right)^{n+\frac{1}{2}} / e = \exp\left\{\int_0^1 \frac{u^2}{(2n+1)^2 - u^2}\, du\right\}$$

$$= \exp\left\{\int_0^1 \frac{u^2}{8m + 1 - u^2}\, du\right\} \qquad \text{(where } m = n(n+1)/2\text{)}$$

$$= \exp\left\{\int_0^1 \frac{\alpha u^2}{1 + \alpha(1 - u^2)}\, du\right\} \qquad \text{(where } \alpha = 1/(8m)\text{)}$$

$$= \exp\left\{\int_0^1 \alpha u^2 - \alpha^2 u^2 (1 - u^2) + \alpha^3 u^2 (1 - u^2)^2 - \alpha^4 u^2 (1 - u^2)^3 + \cdots\, du\right\}$$

$$= \exp\left\{\frac{1}{3}\alpha - \frac{1}{5}\cdot\frac{2}{3}\alpha^2 + \frac{1}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\alpha^3 - \frac{1}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\alpha^4 + \frac{1}{11}\cdot\frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\alpha^5 - \cdots\right\}$$

$$= \exp\left\{\frac{\alpha}{3}\left(1 - \frac{2\alpha}{5}\left(1 - \frac{4\alpha}{7}\left(1 - \frac{6\alpha}{9}\left(1 - \frac{8\alpha}{11}(1 - \cdots)\right)\right)\right)\right)\right\}$$

$$= \exp\left\{\frac{1}{24m}\left(1 - \frac{1}{5\cdot 4m}\left(1 - \frac{2}{7\cdot 4m}\left(1 - \frac{3}{9\cdot 4m}\left(1 - \frac{4}{11\cdot 4m}(1 - \cdots)\right)\right)\right)\right)\right\}.$$

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