2023 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required and it is advisable to leave them off. Each problem has the same point-value.

1. The circumference of Circle O is 34, the circumference of Circle P is 42, and the circumference of Circle T is 53. If one of the three circles is selected at random, find the probability that the radius of that circle is more than 6. Express your answer as a common fraction reduced to lowest terms.

2. In ΔABC, if $AB = BC$, and $\angle BAC = 32°$, by how many degrees does the angle measure of $\angle ABC$ exceed that of $\angle BCA$?

3. For all values of x such that $x \neq 3$, $\frac{5x-15}{12} \div \frac{4x-12}{15} = k$. Find the value of k. Express your answer as a decimal.

4. Three different fruits lie on a table: an orange, an apple, and a banana. If two different fruits are selected at random from among these three, find the probability that the first letter of each fruit selected is a vowel. Express your answer as a common fraction reduced to lowest terms.

5. Two triangles have congruent bases. The heights of each of these two triangles on these two congruent bases are also congruent. Must these two triangles be congruent? For your answer, write the whole word, choosing from "Yes" or "No".

6. If $17.4x + 34.2 = 5.8(3x + y)$, find the value of y. Express your answer as an improper fraction reduced to lowest terms.

7. From all the interior angles of an equiangular pentagon, an equiangular hexagon, and an equiangular heptagon, one of these interior angles is selected at random. Find the probability that the angle selected has a degree measure that is an integer. Express your answer as a common fraction reduced to the lowest terms.

8. In rectangle $RECT$, M is the midpoint of $\overline{RE}$, N is the midpoint of $\overline{EC}$, O is the midpoint of $\overline{TC}$, and P is the midpoint of $\overline{RT}$. If $RP = 6$, and $TO = 17.5$, find the perimeter of quadrilateral $MNOP$.

9. Assume that men and women are adults, and assume that boys and girls are not adults. At a Halloween party, only men, women, boys, and girls attended. There were 17 girls, 11 adults without costumes, 14 women, 10 girls with costumes, 29 people without costumes, 11 women with costumes, and 16 males with costumes. Find the total number of people who attended this Halloween party.

10. Allison walked for a certain number of miles at a constant rate of 4 mph. She then immediately jogged an additional number of miles at a constant rate of 8 mph. She covered a total distance of 29 miles in 7 hours. Find the number of minutes that Allison walked.

11. In $\Delta\ ABC$, the vertices are at $A(4,3)$, $B(14,5)$, and $C(-3,1)$. Find the length of the median from C to $\overline{AB}$. Give your answer in the form $a\sqrt{b}$ where both a and b are integers larger than 1.

12. Given the points $D(1,2)$, $E(10,2)$, $F(9,-2)$. A lattice point is defined as a point for which all coordinates are integers. Find the number of distinct lattice points that are in the interior of ΔDEF.

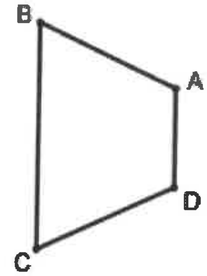
13. When $x^4 - 33$ is divided by $x - 3$, find the remainder.

14. If $43°20'$ were expressed in terms of degrees, the answer would be $43\frac{1}{3}$ degrees. Likewise $43°20'15''$ would be $43\frac{27}{80}$ degrees. By how many degrees does $64\frac{2}{3}$ degrees exceed $60°25'30''$? Express your answer as an improper fraction reduced to lowest terms.

15. If $5x = 7y = z$, and x, y, and z are positive integers, then all of the following must be an integer except:

A) $\frac{z}{xy}$ B) $\frac{z}{7}$ C) $\frac{z}{35}$ D) $\frac{z}{5}$ E) $\frac{y}{5}$

Write the capital letter corresponding to the correct answer on your answer sheet.



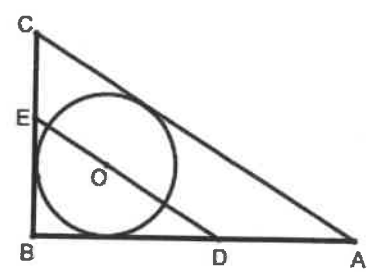
16. In the diagram, $ABCD$ is an isosceles trapezoid with $\overline{BC}$ as one of the bases. $AD = kx + 4$, $DC = 7.6x - 5.1$, $BC = 19x$, and $AB = 3.2x + 21.3$. If k is a positive integer, find the smallest possible value of k.

17. Tiffany was rowing upstream one day when her cap blew off into the stream. She failed to notice it was missing until 18 minutes after it blew off. She immediately turned around and recovered the cap 2.64 miles downstream from where it initially blew off. Assume Tiffany's physical rate of rowing was constant, the rate of the current was constant, and that it took no time to turn around. Find the number of miles per hour in the rate of the current. Express your answer as an exact decimal.

18. A rectangular solid has dimensions of 5, 6, and 8. From one of the faces with smallest area, a circular hole of diameter 4 is drilled at a right angle from the face halfway through to the opposite face. When the circular hole reaches that halfway point, a semicircular hole of diameter 4 is drilled the rest of the way to the opposite face and at a right angle with that opposite face. Find the total surface area of the rectangular solid with the hole. Express your answer as a decimal rounded to the nearest tenth.

19. Two positive integers each of which is less than 10 form a two-digit number. The digits are then switched to form a second two-digit number. When the second two-digit number is subtracted from the original two-digit number, the result is a number that is one less than the second two-digit number. Find the original number.

20. A circle with center at O is inscribed in ΔABC. $AB = 12$, $BC = 5$, and $AC = 13$. Point E lies on $\overline{BC}$, and point D lies on $\overline{AB}$ such that $\overline{DE} \parallel \overline{AC}$ and such that $\overline{DE}$ passes through point O. Find DE. Express your answer as an improper fraction reduced to lowest terms.



Name: ____________________ Team Code: ________

2023 John O'Bryan Mathematical Competition
Freshman/Sophomore Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________

10. ____________________

11. ____________________

12. ____________________

13. ____________________

14. ____________________

15. ____________________

16. ____________________

17. ____________________

18. ____________________

19. ____________________

20. ____________________

Name: ______ANSWERS______

Team Code: ________

2023 John O'Bryan Mathematical Competition
Freshman-Sophomore Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 2 / 3 — Must be this fraction
2. 84 — Degree symbol optional
3. 1.5625 — Must be in decimal form
4. 1 / 3 — Must be this fraction
5. No
6. 171 / 29 — Must be this fraction
7. 11 / 18 — Must be this fraction
8. 74
9. 66
10. 405
11. $3\sqrt{17}$ — Must be in this form
12. 12
13. 48
14. 509 / 120 — Must be this fraction
15. A
16. 5
17. 4.4 — Must be this decimal
18. 314.8 — Must be this decimal
19. 73
20. 221 / 30 — Must be this fraction

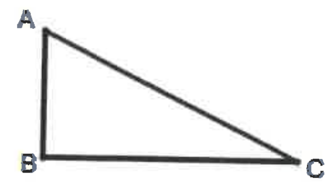
Awards Lists and Solutions to the Team Competition may be found at http://math.nku.edu/job

2023 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Directions: Please answer all questions on the answer sheet provided. All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. **Exact** answers are to be given unless otherwise specified in the question. No units of measurement are required and it is advisable to leave them off. Each problem has the same point-value.

1. Let $i = \sqrt{-1}$.The roots for x of the quadratic equation $x^2 + kx + w = 0$ are $12 - 3i$ and $12 + 3i$. If k and w are integers, find the value of $(2k + 3w)$.

2. In $\triangle ABC$, $AB = 8$, $BC = 15$, and $AC = 17$. A point is selected at random in the interior of line segment $\overline{BC}$. Find the probability that the distance from A to the point selected is more than $2\sqrt{17}$. Express your answer as a common fraction reduced to lowest terms.



3. If $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$ and $x = 2$ and $y = 3$, find the value of z. Express your answer as a decimal.

4. Cindy has three marbles. One is colored red, another white, and the third is green. She has two containers labeled A and B. She selects a marble at random and then chooses a container at random into which she puts the marble. She repeats this process for each of the two remaining marbles. Find the probability that the red marbles was the first marble selected and that the white marble is in the same container as the red marble.

5. Find the 50th term of the arithmetic sequence $5, \frac{24}{5}, \frac{23}{5}, \ldots$. Express your answer as a decimal.

6. If x represents a positive integer, find the sum of all distinct values of x such that $x(x-4)(x+2) \leq 0$.

7. Find the radian period of the graph $y = 3sin(2x)$.

8. Given a sequence $1,4,7,10, \ldots (3n-2)$. If one of the first ten numbers in this sequence is selected at random, find the probability that the number selected is odd. Express your answer as a common fraction reduced to lowest terms.

9. On a flat planar surface, a semicircle is drawn in the exterior of a square with a side of the square as its diameter. If the sum of the areas of the square and the semicircle is 240, find the length of the diameter. Express your answer as a decimal rounded to the nearest hundredth.

10. The distance between two points represented by $(x, 5)$ and $(-3, 26)$ is 29. Find the largest possible value of x.

11. Find the number of years it will take for a sum of money to double if invested at an annual percentage rate of 8.1% and compounded continuously. Express your answer as a decimal rounded to the nearest hundredth of a year.

12. An apartment rental company has 3000 apartments available, and 1900 are presently rented at \$950 per month. The company has decided it will only raise or lower the rent per month on all apartments by integral increments of \$40. A survey has shown that, so long as there are apartments available, for each \$40 drop in rent per month for every apartment, there will be 99 new tenants. Find the number of dollars in the monthly rent that will maximize total income.

13. Let $f(x) = x^2 + 5$ and let $g(x) = kx + w$ where k and w are positive integers. If $f(g(6)) = 1686$, find the possible values of $(k + w)$. For your answer, give the smallest possible value of $(k + w)$.

14. If $(x + y)^5$ is expanded and completely simplified, find the sum of the numerical coefficients.

15. Let $i = \sqrt{-1}$. If the reciprocal of $7 - 2i$ is written in $x + yi$ form where x and y are real numbers, find the value of $(4x + 2y)$. Express your answer as a common fraction reduced to lowest terms.

16. The graph of $y = \frac{2x^3-7x^2+5x-6}{x^2-5x+6}$ has an oblique or slant asymptote of $y = kx + w$. Find the value of $(k + w)$.

17. For all real values of x except for $x = \pm\sqrt{7}$, $\frac{2x^5+x^4+3x^3-x+5}{x^2-7} = 2x^3 + x^2 + 17x + 7 + \frac{kx+w}{x^2-7}$. Find the value of $(k + w)$.

18. On April 1, April puts 1 cent in her piggy bank which was empty. On April 2, April puts 2 additional cents in her piggy bank. On April 3, April puts 4 additional cents in her piggy bank. On April 4, April put 8 additional cents in her piggy bank. She continues this process of doubling the number of cents from the previous day. How many cents will April have in her piggy bank after her deposit on the last day of April?

19. From a 20" × 32" rectangular sheet of cardboard, x" × x" squares are cut from two corners, and x" × 16" rectangles are cut from the other 2 corners. The remaining cardboard is then folded to form a box with a lid. Find the value of x that yields the box of maximum volume. Express your answer as a decimal rounded to the nearest hundredth of an inch.

20. A positive integer, if divided by 12, leaves a remainder of 11; if divided by 11, leave a remainder of 10; if divided by 10, leaves a remainder of 9; if divided by 9, leaves a remainder of 8; if divided by 8, leaves a remainder of 7; if divided by 7, leaves a remainder of 6; if divided by 6, leaves a remainder of 5; and if divided by 5, leaves a remainder of 4. Find the smallest such positive integer.

Name: ____________________ Team Code: ______

2023 John O'Bryan Mathematical Competition
Junior/Senior Individual Test

Note: All answers must be written legibly in the correct blanks on the answer sheet and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

1. ____________________

2. ____________________

3. ____________________

4. ____________________

5. ____________________

6. ____________________

7. ____________________

8. ____________________

9. ____________________

10. ____________________

11. ____________________

12. ____________________

13. ____________________

14. ____________________

15. ____________________

16. ____________________

17. ____________________

18. ____________________

19. ____________________

20. ____________________

Name: ______ANSWERS______

Team Code: ________

2023 John O'Bryan Mathematical Competition
Junior-Senior Individual Test

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value (1 point).

1. 411
2. 13 / 15 — Must be this fraction.
3. 1.2 — Must be this decimal.
4. 1 / 6 — Must be this fraction.
5. – 4.8 — Must be this decimal.
6. 10
7. π
8. 1 / 2 — Must be this fraction.
9. 13.13 — Must be this decimal.
10. 17
11. 8.56 — Must be this decimal.
12. 870
13. 11
14. 32
15. 32 / 53 — Must be this fraction.
16. 5
17. 172
18. 1073741823
19. 4.00
20. 27719

2023 John O'Bryan Mathematical Competition
Questions for the Two-Person Speed Event

*****Calculators may not be used on the first four questions*****

1. Find the sum of the distinct roots of the following four equations:
   $5x - 8 = 112$ $\quad y^2 - 4y - 12 = 0$ $\quad z^2 = 288$ $\quad |w - 1| = 10$

2. Players stand in a circle. Player 1 stays in. Player 2 is knocked out. Player 3, in; Player 4, out. This continues, knocking every other Player out, until only one Player remains. With 9 Players, let k be the number of the last Player remaining; with 11 players, let w be the number of the last Player remaining. Find the value of $(2k + 3w)$.

3. The sum of two numbers is 15 while their product is 5. Let k be the sum of their reciprocals. Let w be the telescoping sum $600 + \sqrt{600 + \sqrt{600 + \sqrt{600 + \cdots}}}$ where the pattern continues indefinitely. Find the value of $k\sqrt{w}$.

4. Let k be the value of the x-coordinate of the center of the circle whose equation is $x^2 + y^2 - 6x + 8y - 24 = 0$. Let w be the value of the y-coordinate of the center of the circle whose equation is $x^2 + y^2 - 16 + 18y - 24x = 0$. Find the value of $(2k + 3w)$.

*****Calculators may be used on the remaining questions*****

5. Let p be the unit's digit of $3^{11182023}$. Let k be an integer greater than 50. One group of 50 numbers has an arithmetic mean of 32. Another group of 70 numbers has an arithmetic mean of k. Let w be the smallest possible value of k such that the arithmetic mean of both groups of numbers together is a positive integer. Find the value of $(w + p)$.

6. Let $\log(k) = 5 - 5\log(2)$. Let w be the number of distinct possible committees of 7 that can be chosen from 5 girls and 4 boys if the majority of members of each committee must be girls. Find the value of $(k + w)$.

7. Working alone at constant rates, it takes the respective number of hours to clean the SU Ballroom: Nora, 3; Katie, 7; Ulrich, 13. With no loss of efficiency, let k and w be the respective number of hours required if the first two work together and then if the last two work together. Expressed as a decimal, find the value of $(k + w)$.

8. Let k be a thirty-two digit number such that the last three digits (hundreds, tens, and units) of k^2 are 225. Let w be the sum of all possibilities for the tens digit of k. Let $i = \sqrt{-1}$. Let p be an integer such that $1 + i$ is a zero of $f(x) = x^3 + px^2 + 32x - 30$. Find the value of $(w + p)$.

T1. Let k be the tangent of the smallest angle of a right triangle whose hypotenuse has a length of 89 and one of whose legs has a length of 39. $[x]$ is defined as the greatest integer which is not greater than x. If $2 < x < 3$, let w be the largest possible integral value of $[x^2] - [x]^2$. Find the value of $(1600k + w)$.

T2. Find the average of the first 50 terms of the arithmetic sequence: 7,10,13,16,.... Express your answer as an exact decimal.

T3. Find the sum of $0.\overline{4}$ (where the "4" repeats) and $0.6\overline{72}$ (where the "72" repeats). Express your answer as an improper fraction reduced to lowest terms.

Names: ______________________________ **School:** ______________________

2023 John O'Bryan Mathematical Competition
Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

		SCORE
1.	______________________________	_____
2.	______________________________	_____
3.	______________________________	_____
4.	______________________________	_____
5.	______________________________	_____
6.	______________________________	_____
7.	______________________________	_____
8.	______________________________	_____
T1.	______________________________	
T2.	______________________________	

Calculators are not allowed to be used on the first four questions!

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

Names: ______________________ School: ____________________

2023 John O'Bryan Mathematical Competition
Answers for the Two-Person Speed Event

Note: All answers must be written legibly and in simplest form. Exact answers are to be given unless otherwise specified in the question. No units of measurement are required. Each problem has the same point-value.

Calculators are not allowed to be used on the first four questions!

	Answer	SCORE
1.	30	____
2.	27	____
3.	75	____
4.	– 21	____
5.	63	____
6.	3151	____
7.	6.65 **Must be this decimal**	____
8.	1	____
T1.	784	
T2.	80.5 **Must be this decimal**	
T3.	553/49	

This competition consists of eight competitive rounds. Correct answers will receive the following scores:

1st: 7 points
2nd: 5 points
All Others: 3 points

There is a three minute time limit on each round. You may submit only one answer each round. To submit your answer, fold this sheet **lengthwise** and hold it high in the air so that a proctor may check your answer.

SCORE

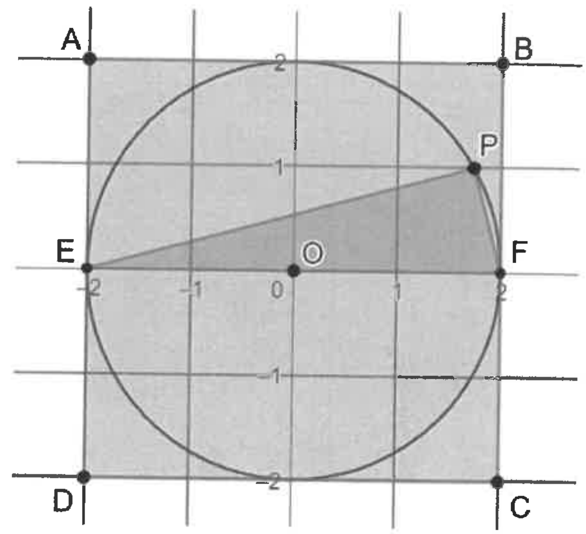
2023 John O'Bryan Mathematics Competition
5-Person Team Test

Abbreviated Instructions: Answer each of the following questions using separate sheet(s) of paper. Questions will not be scored without the following two items:

- Place your team code in the upper right corner of each page that will be turned in.
- Place question numbers in the upper left corner of each page that will be turned in.

Questions are equally weighted. **Teams must show complete solutions (not just answers) to receive credit.** More specific instructions are read verbally at the beginning of the test.

1. In the figure, quadrilateral ABCD is square and inscribed circle O has a radius of 2 units. The x-coordinate of point P is $\sqrt{3}$.



   a. Find the area of triangle EPF.

   b. Find the sum of the squares of the distances from point P to each vertex of the square.

   c. Does the answer to part (b) depend upon the location of point P or will you obtain the same sum of squares for any point on the circle? Justify your answer.

2. On day 1, Paris writes the number 1. On day 2, she writes the numbers 2 and 3. On day 3, she writes the numbers 4, 5, and 6. Paris continues writing numbers in this way, writing N numbers on the Nth day. **Remember for all problems you must show or explain how you obtained your answers!**

   a. What is the largest number that Paris writes on the 20th day?
   b. What is the sum of the numbers that Paris writes on the 20th day?
   c. What is the sum of the numbers that Paris writes on the Nth day?

3. For each positive integer n, let f_n and g_n be the positive integers such that

$$\left(\sqrt{3}+\sqrt{2}\right)^{2n} = f_n + g_n\sqrt{6} \text{ and } \left(\sqrt{3}-\sqrt{2}\right)^{2n} = f_n - g_n\sqrt{6}$$

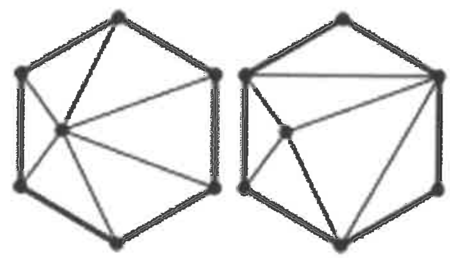
   a. Determine the values of f_2 and g_2.

   b. Show that $2f_n - 1 < \left(\sqrt{3}+\sqrt{2}\right)^{2n} < 2f_n$ for all positive integers n

   c. Let d_n be the ones (or units) digit of the number $\left(\sqrt{3}+\sqrt{2}\right)^{2n}$ when it is written in decimal form. Determine the value of $d_1 + d_2 + d_4$

4. Complete each of the following. **Remember for all problems you must show or explain how you obtained your answers!**
   a. Determine the positive integer c for which $\frac{1}{4} - \frac{1}{c} = \frac{1}{6}$
   b. Determine the number of pairs of positive integers (d, e) for which $\frac{1}{d} - \frac{1}{e} = \frac{1}{12}$
   c. Show that for every prime number p, there are at least two pairs (r, s) of positive integers for which $\frac{1}{r} - \frac{1}{s} = \frac{1}{p^2}$

5. A triangulation of a regular polygon is a division of its interior into triangular regions. In a triangulation, each vertex of each triangle is either a vertex of the regular polygon or an interior point of the polygon.



T(6,1) = 6

The number of triangles formed by a triangulation of a regular polygon with n vertices and k interior points is a constant and is denoted $T(n,k)$. For example, if $n = 6$ and $k = 1$, then $T(6,1) = 6$ as is illustrated in the diagram.

Remember for all problems you must show or explain how you obtained your answers!

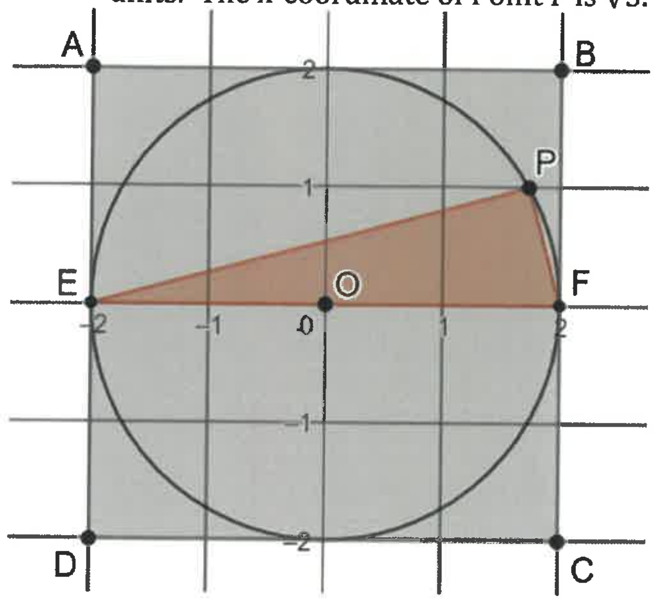
   a. Determine the value of T(3,2).
   b. Determine the value of T(4,100).
   c. Determine the value of n for which T(n,n) = 1000

6. Define the function $f(x) = \frac{x}{x-1}$ for $x \neq 1$. **Remember for all problems you must show or explain how you obtained your answers!**

   a. Determine all real numbers $r \neq 1$ for which $f(r) = r$
   b. Find $f(f(x))$, where $x \neq 1$
   c. Suppose k is a real number and define $g(x) = \frac{2x}{x+k}$ for $x \neq (-k)$. Determine all real values of k for which $g(g(x)) = x$, where $g(x) \neq (-k)$.

John O'Bryan 2023 Team Test Key

1. In the figure below, quadrilateral ABCD is square and inscribed circle O has a radius of 2 units. The x-coordinate of Point P is $\sqrt{3}$.



a. Find the area of triangle EPF

The equation of the circle is $x^2 + y^2 = 4$. If the x-coordinate of P is $\sqrt{3}$, then $(\sqrt{3})^2 + y^2 = 4$. So, $y = 1$. With a base of 4 and altitude of 1, the area of triangle EPF is $(1/2)(4)(1) = 2$ square units.

b. Find the sum of the squares of the distances from point P to each vertex of the square.

Using the distance formula:

$PA^2 = \left(\sqrt{3} - (-2)\right)^2 + (1 - 2)^2 = 8 + 4\sqrt{3}$

$PB^2 = \left(\sqrt{3} - 2\right)^2 + (1 - 2)^2 = 8 - 4\sqrt{3}$

$PC^2 = \left(\sqrt{3} - 2\right)^2 + (1 - (-2))^2 = 16 - 4\sqrt{3}$

$PD^2 = \left(\sqrt{3} - (-2)\right)^2 + (1 - (-2))^2 = 16 + 4\sqrt{3}$

So, $PA^2 + PB^2 + PC^2 + PD^2 = 48$

c. Does the answer to part (b) depend upon the location of point P or will you obtain the same sum of squares for any point on the circle? Justify your answer.

No, the sum of the squares remains the same (48) for any point on the circle. To show this is true for any point on the top half of the circle (the case for points on the lower half of the circle is the same), use the fact that points on the top half have coordinates (x, $\sqrt{4-x^2}$). Using the distance formula and this general point:

$PA^2 = (x+2)^2 + \left(\sqrt{4-x^2} - 2\right)^2$
$= x^2 + 4x + 4 + 4 - x^2 - 4\sqrt{4-x^2} + 4$
$= 12 + 4x - 4\sqrt{4-x^2}$

$PB^2 = (x-2)^2 + \left(\sqrt{4-x^2} - 2\right)^2$
$= x^2 - 4x + 4 + 4 - x^2 - 4\sqrt{4-x^2} + 4$
$= 12 - 4x - 4\sqrt{4-x^2}$

$PC^2 = (x-2)^2 + \left(\sqrt{4-x^2} + 2\right)^2$
$= x^2 - 4x + 4 + 4 - x^2 + 4\sqrt{4-x^2} + 4$
$= 12 - 4x + 4\sqrt{4-x^2}$

$PD^2 = (x+2)^2 + \left(\sqrt{4-x^2} + 2\right)^2$
$= x^2 + 4x + 4 + 4 - x^2 + 4\sqrt{4-x^2} + 4$
$= 12 + 4x + 4\sqrt{4-x^2}$

So, So, $PA^2 + PB^2 + PC^2 + PD^2 = 48$

2. On day 1, Paris writes the number 1.
On day 2, she writes the numbers 2 and 3.
On day 3, she writes the numbers 4, 5, and 6.
Paris continues writing numbers in this way, writing N numbers on the Nth day.

a. What is the largest number that Paris writes on the 20th day? Show or explain how you obtained your answer.

From the pattern, the largest number written on day N equals the sum of the first n natural numbers $= \frac{N(N+1)}{2}$. For instance, the largest number written on day 1 $= \frac{1(1+1)}{2} = 1$, the largest number written on day 2 $= \frac{2(2+1)}{2} = 3$, the largest number written on day 3 $= \frac{3(3+1)}{2} = 6$, and so on. So, the largest number written on day 20 $= \frac{20(20+1)}{2} = 210$.

b. What is the sum of the numbers that Paris writes on the 20th day? Show or explain how you obtained your answer.

The largest number written on day 19 $= \frac{19(19+1)}{2} = 190$, so the first number written on day 20 is 191. Thus, the sum of numbers written on day 20 equals 191 + 192 + ... + 210. These can either be added directly, or it can be noticed that these 20 numbers represent an arithmetic sum, with $a_n = 191 + (n - 1)$ with n = 1, 2, ..., 20. The sum of the first 20 numbers of this arithmetic sum equals $\frac{20(191+210)}{2} = 4010$

c. What is the sum of the numbers that Paris writes on the Nth day? Justify your answer.

Generalizing from the solution to part (b), the largest number written on the (N – 1)st day $= \frac{(N-1)(N)}{2}$. So, the first number written on the Nth day $= \frac{(N-1)(N)}{2} + 1$. Also, the last number written on the Nth day $= \frac{N(N+1)}{2}$. As in part (b), these numbers can be represented by an arithmetic sum: $a_n = \frac{(N-1)(N)}{2} + 1 + (n-1)$, with n = 1, 2, ..., N. The sum of the first N numbers of this arithmetic sum equals $\frac{N(\frac{(N-1)(N)}{2}+1+\frac{N(N+1)}{2})}{2} = \frac{N(N^2+1)}{2}$

3. For each positive integer n, let f_n and g_n be the positive integers such that

$$(\sqrt{3}+\sqrt{2})^{2n} = f_n + g_n\sqrt{6} \text{ and } (\sqrt{3}-\sqrt{2})^{2n} = f_n - g_n\sqrt{6}$$

a. Determine the values of f_2 and g_2.

Since $(\sqrt{3}+\sqrt{2})^{2n} = f_n + g_n\sqrt{6}$, $(\sqrt{3}+\sqrt{2})^4 = f_2 + g_2\sqrt{6}$.

Since $(\sqrt{3}+\sqrt{2})^2 = 3 + 2\sqrt{6} + 2 = 5 + 2\sqrt{6}$, $(\sqrt{3}+\sqrt{2})^4 = \left((\sqrt{3}+\sqrt{2})^2\right)^2 = (5+2\sqrt{6})^2 = 25 + 20\sqrt{6} + 24 = 49 + 20\sqrt{6}$. So, $f_2 = 49$ and $g_2 = 20$.

b. Show that $2f_n - 1 < (\sqrt{3}+\sqrt{2})^{2n} < 2f_n$ for all positive integers n.

Since $(\sqrt{3}+\sqrt{2})^{2n} = f_n + g_n\sqrt{6}$ and $(\sqrt{3}-\sqrt{2})^{2n} = f_n - g_n\sqrt{6}$,

$f_n + g_n\sqrt{6} + f_n - g_n\sqrt{6} = (\sqrt{3}+\sqrt{2})^{2n} + (\sqrt{3}-\sqrt{2})^{2n}$

So, $2f_n = (\sqrt{3}+\sqrt{2})^{2n} + (\sqrt{3}-\sqrt{2})^{2n}$

$\sqrt{3}-\sqrt{2} \approx 1.732 - 1.414 < 1$. So, $0 < (\sqrt{3}-\sqrt{2})^{2n} < 1$ for all positive integers n.
Adding, $(\sqrt{3}+\sqrt{2})^{2n}$ to each term of this inequality yields $(\sqrt{3}+\sqrt{2})^{2n} < (\sqrt{3}+\sqrt{2})^{2n} + (\sqrt{3}-\sqrt{2})^{2n} < (\sqrt{3}+\sqrt{2})^{2n} + 1$, or $(\sqrt{3}+\sqrt{2})^{2n} < 2f_n < (\sqrt{3}+\sqrt{2})^{2n} + 1$

So, $(\sqrt{3}+\sqrt{2})^{2n} < 2f_n$ and $2f_n - 1 < (\sqrt{3}+\sqrt{2})^{2n}$, which is the desired result.

c. Let d_n be the ones (or units) digit of the number $(\sqrt{3}+\sqrt{2})^{2n}$ when it is written in decimal form. Determine the value of $d_1 + d_2 + d_4$.

Using the same logic as part (a), $f_1 = 5$ and $g_1 = 2$.
From part (b), $2f_1 - 1 < (\sqrt{3}+\sqrt{2})^2 < 2f_1$, or $9 < (\sqrt{3}+\sqrt{2})^2 < 10$. So, $d_1 = 9$.

Similarly, $2f_2 - 1 < (\sqrt{3}+\sqrt{2})^4 < 2f_2$, or $97 < (\sqrt{3}+\sqrt{2})^4 < 98$. So, $d_2 = 7$.

Since $(\sqrt{3}+\sqrt{2})^4 = 49 + 20\sqrt{6}$, $(\sqrt{3}+\sqrt{2})^8 = \left((\sqrt{3}+\sqrt{2})^4\right)^2 = (49+20\sqrt{6})^2 = 2401 + 1960\sqrt{6} + 2400 = 4801 + 1960\sqrt{6}$. So, $f_4 = 4801$ and $g_2 = 1960$.

Since $2f_4 - 1 < (\sqrt{3}+\sqrt{2})^8 < 2f_4$, or $9601 < (\sqrt{3}+\sqrt{2})^8 < 9602$. So, $d_4 = 1$.

Thus, $d_1 + d_2 + d_4 = 17$

4. a. Determine the positive integer c for which $\frac{1}{4} - \frac{1}{c} = \frac{1}{6}$

Multiply the equation by 12c, so that $3c - 12 = 2c$. So, $c = 12$

b. Determine the number of pairs of positive integers (d, e) for which $\frac{1}{d} - \frac{1}{e} = \frac{1}{12}$

Multiply the equation by 12de to obtain $12e - 12d = de$.
So:
$0 = de + 12d - 12e$
$0 = d(e + 12) - 12e$
$-144 = d(e + 12) - 12e - 144$
$-144 = d(e + 12) - 12(e + 12) = (d - 12)(e + 12)$

Since $e > 0$, $e + 12 > 0$ and $d - 12 < 0$ (since the product is -144). Using a table of values, there **are 7 possible pairs of integers.**

d-12	e+12	d	e
−1	144	11	132
−2	72	10	60
−3	48	9	36
−4	36	8	24
−6	24	6	12
−8	18	4	6
−9	16	3	4

c. Show that for every prime number p, there are at least two pairs (r, s) of positive integers for which $\frac{1}{r} - \frac{1}{s} = \frac{1}{p^2}$

Multiply the equation by rsp^2 to obtain $sp^2 - rp^2 = rs$. So:
$0 = rs + rp^2 - sp^2$
$0 = r(s+p^2) - sp^2$
$-p^4 = r(s+p^2) - sp^2 - p^4$
$-p^4 = r(s+p^2) - p^2(s + p^2)$
$-p^4 = (r - p^2)(s+p^2)$

Since r and s are positive, $r - p^2 < 0$ and $s + p^2 > 0$ are a divisor pair of $-p^4$, with $-p^2 < r - p^2 < 0$.

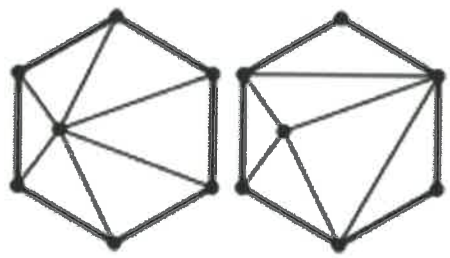
If $r - p^2 = -1$, then $s + p^2 = p^4$, yielding the pair $r = p^2 - 1$ and $s = p^4 - p^2$.

If $r - p^2 = -p$, then $s + p^2 = p^3$, yielding the pair $r = p^2 - p$ and $s = p^3 - p^2$.

Since p is prime, these are two pairs of distinct positive integers.

5. A triangulation of a regular polygon is a division of its interior into triangular regions. In a triangulation, each vertex of each triangle is either a vertex of the regular polygon or an interior point of the polygon.

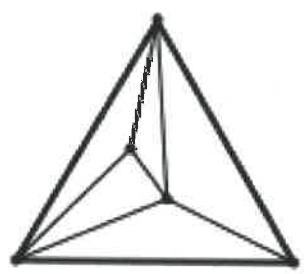
The number of triangles formed by a triangulation of a regular polygon with n vertices and k interior points is a constant and is denoted T(n,k). If n = 6 and k = 1, for instance, T(6,1) = 6, as is illustrated below:



T(6,1) = 6

a. Determine the value of T(3,2).

Since T(3,2) is a constant, all that is needed is a sketch of the region, as below. From this sketch, T(3,2) = 5



b. Determine the value of T(4,100).

Observe the pattern from the quadrilaterals below:

T(4,0) = 2

If the first interior point is added in one of the T(4,0) triangles, T(4,0) is reduced by one triangle and three new triangles are added, with the net result of PLUS two triangles. So, T(4,1) = 4. Note, the result is the same if the point is added on the boundary of an interior triangle.

If an interior point is added to one of the T(4,1) triangles, T(4,1) is reduced by one triangle and three new triangles are added (as below), with the net result of plus two triangles. So, T(4,2) = 6

Overall, the pattern is T(4,k) = 2(k+1). **A proof by induction would provide additional points!**

So, T(4,100) = 2(100 + 1) = 202 triangles.

c. Determine the value of n for which T(n,n) = 1000

In the triangulation of a regular n-gon with no interior points, we can choose any of the n-vertices and connect this vertex to each of the n – 3 non-adjacent vertices. This created n – 2 triangles, so T(n,0) = n – 2.

The reasoning in part b can be extended to n-gons. Any additional interior point must be within (or on the boundary) of an existing triangle. So, T(n,k) is reduced by one triangle and three new triangles are added, with the net result of plus two triangles. This leads to the following pattern:

T(n,0) = n – 2
T(n,1) = (n – 2) + 2
T(n,2) = ((n – 2) + 2) + 2 = (n – 2) + 2(2)
T(n,3) = ((n – 2) + 2(2)) + 2 = (n – 2) + 3(2)
Overall, T(n,k) = (n – 2) + 2k.

Let T(n,n) = (n – 2) + 2n = 3n – 2 = 1000
So, 3n = 1002 and n = 334 interior points.

6. Define the function $f(x) = \frac{x}{x-1}$ for $x \neq 1$.

a. Determine all real numbers $r \neq 1$ for which $f(r) = r$

If $f(r) = r$, then $\frac{r}{r-1} = r$

Since $r \neq 1$, $r = r(r-1)$ or $r = r^2 - r$.
Thus, $r^2 - 2r = r(r-2) = 0$, so $r = 0$ or $r = 2$.

b. Find $f(f(x))$, where $x \neq 1$

Since $f(x) = \frac{x}{x-1}$, $f(f(x)) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = \frac{x}{x-(x-1)} = x$

c. Suppose k is a real number and define $g(x) = \frac{2x}{x+k}$ for $x \neq -k$. Determine all real values of k for which $g(g(x)) = x$, where $g(x) \neq -k$.

If $x \neq -k$, then $g(x)$ is defined for all other real values.

If $x \neq -k$ and $g(x) \neq -k$, then $g(g(x)) = \frac{2\left(\frac{2x}{x+k}\right)}{\frac{2x}{x+k}+k}$ is defined for all other real values.

Let $g(g(x)) = \frac{2\left(\frac{2x}{x+k}\right)}{\frac{2x}{x+k}+k} = x$.

Then, $\frac{4x}{2x+k(x+k)} = x$

So: $4x = x(2x + k(x + k))$
$4x = x(2x + kx + k^2)$
$4x = 2x^2 + kx^2 + k^2x$
$0 = (2 + k)x^2 + (k^2 - 4)x$
$0 = (k + 2)[x2 + (k - 2)x]$

If **$k = -2$**, $\frac{4x}{2x+k(x+k)} = x$ for all values of x for which the function is defined (i.e., $x \neq -k$, and $g(x) \neq -k$)