MAT225 Section Summary: 1.3

Vector Equations

1. Definitions

- **vector**

  a matrix with only a single column ("column vector"). The entries are called the **components** of the vector.

- **zero vector**

  the vector whose components are all 0.

- **one vector** (not a unit vector!)

  the vector whose components are all 1.

- **scalar multiple** of a vector

  a product of a constant ("**scalar**") and a vector, the operation being carried out component-wise.

- **vector sum**: the vector created by adding two vectors, the sums being carried out component-wise. The sum of vectors can be found using the "parallelogram rule": the butt of vector $\mathbf{v}_2$ is placed at the tip of the vector $\mathbf{v}_1$, and the vector from the butt of $\mathbf{v}_1$ to the tip of $\mathbf{v}_2$ is the sum.

- **linear combination** of vectors

  any sum of vectors scaled by coefficients.

  Example: #4, p. 37

- **span**

  the span of a set of vectors $\{\mathbf{v}_1, ...\mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors that can be solutions of the system

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

linear combination

Q: What is the geometry of a span? What cases should be considered?

2. Properties/Tricks/Hints/Etc.

- The vector equation

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

has the same solution as the linear system whose augmented matrix is

$$[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \ldots \ \ \mathbf{v}_p \ \ \mathbf{b}]$$

Example: #9, p. 37

#9

$$\begin{aligned} x_2 + 5x_3 &= 0 \\ 4x_1 + 6x_2 - x_3 &= 0 \\ -x_1 + 3x_2 - 8x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 5 & 0 \\ 4 & 6 & -1 & 0 \\ -1 & 3 & -8 & 0 \end{bmatrix}$$

$$\bar{a}_1 = \begin{bmatrix} 0 \\ 4 \\ -1 \end{bmatrix} \quad \bar{a}_2 = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix} \quad \bar{a}_3 = \begin{bmatrix} 5 \\ -1 \\ -8 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example: #12, p. 38

#12

$$\bar{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \quad \bar{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix} \quad \bar{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$$

$$c_1\bar{a}_1 + c_2\bar{a}_2 + c_3\bar{a} = \bar{b}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & -5 \\ -2 & 5 & 0 & 11 \\ 2 & 5 & 8 & -7 \end{array}\right]$$

- Two vectors are equal only if they have the same dimensions, and their components are the same.

- Algebraic properties of the **vector space** $\mathbb{R}^n$: for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d,

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — commutativity
(b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — associativity
(c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$ — existence of a 0
(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ — inverses
(e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — distributivity
(f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — "
(g) $c(d\mathbf{u}) = (cd)\mathbf{u}$ — associativity
(h) $1\mathbf{u} = \mathbf{u}$ — existence of unit in the constants

3. Summary

Vectors provide a wonderful way for us to write systems of equations compactly. You should already be familiar with two-d and three-d vectors from calculus classes. We now want to extend notions from those spaces into n-dimensional space. For example, vector addition is carried out component-wise.

The interesting new concept introduced in this section is that of **span**: roughly, the span of a set of vectors $\{\mathbf{v}_1, \ldots \mathbf{v}_p\}$ is the subspace generated by linear combinations of the vectors $\mathbf{v}_i$. The span represents the set of vectors that can be solutions of the system

$$a_1\mathbf{v}_1 + \ldots + a_p\mathbf{v}_p = \mathbf{b}$$

Example: #21, p. 38

#17 $\quad \bar{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} \quad \bar{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} \quad \bar{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$

Example: #27, p. 38

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \end{bmatrix} \Rightarrow y = -3, \quad x = -2$$

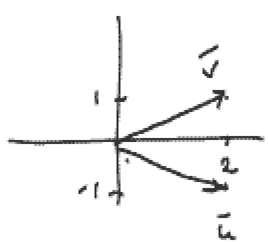
where $(x\bar{a}_1 + y\bar{a}_2 = \bar{b})$

over determined

3 constraints in 2 unknowns $\Rightarrow -2 \cdot -2 + (-3)\cdot 7 = h$

$$-17 = h$$

#2 $\quad \bar{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ + $\bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in the span of $\bar{u}$ + $\bar{v}$ for any choice of h + k.



$$\left.\begin{aligned} 2x + 2y &= h \\ -x + y &= k \end{aligned}\right\} \text{ return to H.S.}$$

$$\begin{bmatrix} 2 & 2 & h \\ -1 & 1 & k \end{bmatrix}$$ row reduce + conclude unique soln.