MAT225 Section Summary: 1.4

The Matrix Equation $A\mathbf{x} = \mathbf{b}$

1. Definitions

- **product of matrix A and vector $\mathbf{x}$**

  If A is an m x n matrix, with columns $\mathbf{a}_1, \mathbf{a}_2 \ldots, \mathbf{a}_n$, and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the product of A and $\mathbf{x}$ is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

$$\overset{430}{} \quad 2000 \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \vdots \\ \bar{x} \end{bmatrix} = \begin{bmatrix} \bar{b} \end{bmatrix} 2000$$

- **identity matrix**

  a matrix with 1's on the diagonal (top left to bottom right), and 0 everywhere else.

$$I_5 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$I \cdot \bar{x} = \bar{x} \qquad A_{m \times n} \, \bar{x}_{n \times 1} = \bar{b}_{m \times 1}$$

2. Theorems/Formulas

Theorem Four (p. 43): Let A be an m x n matrix. Then the following statements are logically equivalent. That is, for a particular A, either they are all true statements or they are all false.

(a) For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

(b) Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Theorem Five (p. 45): If A is an m x n matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$

3. Properties/Tricks/Hints/Etc.

Row-Vector rule for computing $A\mathbf{x}$:

If the product $A\mathbf{x}$ is defined, then the *ith* entry in the vector $A\mathbf{x}$ (yes, it's a vector!) is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

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4. Summary

Once again, we yet another representation for a system of linear equations – my god, will it never end? This is the last we'll examine, and probably the most important. Theorem four pulls all these forms together. Spans, pivots, linear combinations, matrix equations collide!

Matrix/vector multiplication is defined. One form that I find particularly useful is the so-called "row-vector rule": a row of the matrix slams into the variable vector **x**, to produce a single entry in the **b** vector.

#4, 8, 9

#4

a) $\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot\begin{bmatrix} 8 \\ 5 \end{bmatrix} + 1\cdot\begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} 8 & 3 & -4 \\ 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 8 & 3 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 5 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 8+3+-4 \\ 5+1+2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

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$$b_i = \sum_{j=1}^{n} a_{ij}\cdot x_j$$

where

$$A\bar{x} = \begin{array}{cc} \begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{bmatrix} & i^{th} \text{ row} \\ j^{th} \text{ column} & \\ m \times n & \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}$$