MAT225 Section Summary: 1.5

Solution Sets of Linear Systems

1. Definitions

$A\bar{x} = \bar{b}$, replace $\bar{b}$ by $\bar{0}$ (vector!)

- **homogeneous system**

  a system of linear equations of the form $A_{m \times n}\mathbf{x} = \mathbf{0}_{m \times 1}$. This system always has at least one solution: the $\mathbf{0}_{n \times 1}$ vector, called the **trivial solution.** Other solutions are called **nontrivial solutions.**

2. Theorems/Formulas

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system of equations has at least one free variable.

Theorem 6: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given vector $\mathbf{b}$, and let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof: #25, p. 56

Given $\bar{p}$, soln to $A\bar{x} = \bar{b}$, + $\bar{v}_h$ soln to $A\bar{x} = \bar{0}$

Call
$\bar{\omega} = \bar{p} + \bar{v}_h$
$A\bar{\omega} = A\bar{p} + A\bar{v}_h$
$= \bar{b} + \bar{0}$
$= \bar{b}$

3. Properties/Tricks/Hints/Etc.

Observe that

$$A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = \mathbf{b}$$

This shows the most general form of the solution of the matrix system $A\mathbf{x} = \mathbf{b}$.

Writing a solution set (of a consistent system) in parametric vector form:

(a) Row reduce the augmented matrix to reduced echelon form.

and

$$\langle a_{21}, a_{22}, a_{23}\rangle \cdot \langle x_1, x_2, x_3\rangle = 0$$

i.e., that the $\mathbf{x}$ is orthogonal to both row vector ($\mathbf{A}_1$ and $\mathbf{A}_2$).

Now if

$$[a_{11} a_{12} a_{13}] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [b]$$

this says that

$$\langle a_{11}, a_{12}, a_{13}\rangle \cdot \langle x_1, x_2, x_3\rangle = b.$$

That is, that the projection of $\mathbf{x}$ onto $\mathbf{A}_1$ is equal to b

You remember what this means: that

$$\mathbf{A}_1 \cdot \mathbf{x} = |\mathbf{A}_1||\mathbf{x}|\cos(\theta)$$

where θ is the angle between the vectors. Hence

$$A\mathbf{x} = \mathbf{b}$$

says: "the projections of $\mathbf{x}$ onto the rows of A make up the components of $\mathbf{b}$", and if

$$A\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is orthogonal to every row of A; or, alternatively

"$\mathbf{x}$ is orthogonal to the span of the row vectors of A".

The bang is still this:

the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

#25, p 56

Part a) We just did that:

$$A\bar{w} = A(\bar{p} + \bar{v}_h) = A\bar{p} + A\bar{v}_h = \bar{b} + \bar{0} = \bar{b}$$

Part b) Given $\bar{w}$ / $A\bar{w} = \bar{b}$ — can we express it as $\bar{w} = \bar{p} + \bar{v}_h$?

Consider the vector

$$\underline{u} = \bar{w} - \bar{p}$$

$$A\bar{u} = A(\bar{w} - \bar{p}) = A\bar{w} - A\bar{p} = \bar{b} - \bar{b} = \bar{0}$$

So $\bar{u}$ is a soln. of $A\bar{x} = \bar{0}$. Hence

$$\bar{w} = \bar{p} + \bar{u},$$

of the form $\bar{p} + \bar{v}_h$.

What does $A\bar{x} = \bar{0}$ say?

row vector

$$\begin{bmatrix} \overline{\qquad\qquad} \\ A \\ \\ \\ \end{bmatrix} \begin{bmatrix} \bar{x} \\ \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{0} \\ \\ \\ \end{bmatrix}$$

$$b_i = \sum_{j=1}^{n} A_{ij}\, x_j = \langle A_{i1}, A_{i2}, \dots, A_{in} \rangle \cdot \langle x_1, x_2, \dots, x_n \rangle$$

"Calculus" notation

Defn of dot product

If $b_i = 0$, it means that the two vectors – the row of A & the vector $\bar{x}$ – are orthogonal – mutually perpendicular.

So a non-trivial soln $\bar{x}$ of $A\bar{x} = \bar{0}$ has the property that it's orthogonal to every row of A. Neat!