MAT225 Section Summary: 1.5
Solution Sets of Linear Systems

1. **Definitions**

- **homogeneous system**

  a system of linear equations of the form $A_{m \times n}\mathbf{x} = \mathbf{0}_{m \times 1}$. This system always has at least one solution: the $\mathbf{0}_{n \times 1}$ vector, called the **trivial solution.** Other solutions are called **nontrivial solutions.**

2. **Theorems/Formulas**

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the system of equations has at least one free variable.

Theorem 6: Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given vector $\mathbf{b}$, and let $\mathbf{p}$ be a particular solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Proof: #25, p. 56

3. Properties/Tricks/Hints/Etc.

Observe that

$$\boxed{A\mathbf{w} = A(\mathbf{p} + \mathbf{v}_h) = \mathbf{b}}$$

This shows the most general form of the solution of the matrix system $A\mathbf{x} = \mathbf{b}$.

Writing a solution set (of a consistent system) in parametric vector form:

(a) Row reduce the augmented matrix to reduced echelon form.

and

$$\langle a_{21}, a_{22}, a_{23}\rangle \cdot \langle x_1, x_2, x_3\rangle = 0$$

i.e., that the $\mathbf{x}$ is orthogonal to both row vector ($\mathbf{A}_1$ and $\mathbf{A}_2$).

Now if

$$[a_{11}a_{12}a_{13}]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [b]$$

this says that

$$\langle a_{11}, a_{12}, a_{13}\rangle \cdot \langle x_1, x_2, x_3\rangle = b.$$

That is, that the projection of $\mathbf{x}$ onto $\mathbf{A}_1$ is equal to b

You remember what this means: that

$$\mathbf{A}_1 \cdot \mathbf{x} = |\mathbf{A}_1||\mathbf{x}|\cos(\theta)$$

where θ is the angle between the vectors. Hence

$$A\mathbf{x} = \mathbf{b}$$

says: "the projections of $\mathbf{x}$ onto the rows of A make up the components of $\mathbf{b}$", and if

$$A\mathbf{x} = \mathbf{0}$$

then $\mathbf{x}$ is orthogonal to every row of A; or, alternatively

"$\mathbf{x}$ is orthogonal to the span of the row vectors of A".

The bang is still this:

the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} \\ A \\ \\ \end{bmatrix}_{m\times n} \begin{bmatrix} \bar{x} \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \\ 0 \end{bmatrix}_{m\times 1}$$

$$b_i = \langle A_{i1}, \ldots A_{in}\rangle \cdot \langle x_1, \ldots, x_n\rangle$$

$$= 0 \Rightarrow$$

two vectors are orthogonal (perpendicular)

Consider two solns to $A\bar{x} = \bar{0}$, $\bar{v}_1 + \bar{v}_2$:

then any linear combination is a solution!

$$A(\alpha \bar{v}_1 + \beta \bar{v}_2) = \alpha A \bar{v}_1 + \beta A \bar{v}_2$$
$$= \alpha \bar{0} + \beta \bar{0} = \bar{0}$$

#6, p55

$$x_1 + 3x_2 - 5x_3 = 0$$
$$x_1 + 4x_2 - 8x_3 = 0$$
$$-3x_1 - 7x_2 + 9x_3 = 0$$

$$\begin{bmatrix} 1 & 3 & -5 & 0 \\ 1 & 4 & -8 & 0 \\ -3 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -5 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{l} x_1 + 4x_3 = 0 \\ x_2 - 3x_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = -4x_3 \\ x_2 = 3x_3 \end{array}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ 3x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

Soln of the homogeneous equation

What if we'd ended on

$$\sim \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 + 4x_3 = 0 \Rightarrow x_1 = -4x_3$$

Soln would have been

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

#35 Construct $A_{3\times 3}$ / $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a soln of

$$A\bar{x} = \bar{0}.$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad I \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$