MAT225 Section Summary: 1.7
Linear Independence

Summary

This section offers a different take on the equation $A\mathbf{x} = \mathbf{0}$ that we studied in section 1.5: we focus on the columns of A, and ask what relation must exist between them when a non-trivial solution of the homogeneous system exists. If we think of the matrix A in terms of its column vectors (call them $\mathbf{a}_i$), then $A\mathbf{x} = \mathbf{0} \iff$

$$\begin{bmatrix} A \end{bmatrix}\begin{bmatrix} \bar{x} \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$$

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{0} \qquad (1)$$

Definition: a set of vectors $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is **linearly independent** if and only if (??) has only the trivial solution (that is, $x_i = 0 \forall i$).

The zero vector $\bar{x} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$ is a soln (trivial)

Translation: The columns of A are linearly independent $\iff$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Definition: if (??) has a non-trivial solution, then the set of vectors $S = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ is **linearly dependent**.

The obvious consequence, contained in theorem 7, is that one of the vectors of S can be expressed as a linear combination of the others. From (??) we deduce that one of the coefficient x_i is non-zero: if $x_i \neq 0$, then we can solve for $\mathbf{a}_i$ as

$$\mathbf{a}_i = \frac{-1}{x_i}\left(x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_{i-1}\mathbf{a}_{i-1} + x_{i+1}\mathbf{a}_{i+1} + \ldots + x_n\mathbf{a}_n\right)$$

It's pretty clear that if S contains the zero vector, then the set is linearly dependent, since we can set the coefficient of the zero vector to 1, set all the other coefficients to zero, and we have a non-trivial solution to the homogeneous equation (Theorem 8).

It should also be clear from our discussions of spans that if we have more vectors in the set than the size of the space in which the vectors reside then the set will be linearly dependent. For example, if you have four distinct vectors in three-space, then the set will be linearly dependent. This is the substance of theorem 8.

#8 p 71

$$A = \begin{bmatrix} 1 & -3 & 3 & -2 \\ -3 & 7 & -1 & 2 \\ 0 & 1 & -4 & 3 \end{bmatrix}$$; can I write

$$x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3 + x_4\bar{a}_4 = \bar{0}$$

for some nonzero x_i ?

We have four vectors in 3-space — can one be written as a linear combination of the others? Absolutely.

These vectors do not form a linearly independent set.

#1 $\begin{bmatrix}5\\0\\0\end{bmatrix}, \begin{bmatrix}7\\2\\-6\end{bmatrix}, \begin{bmatrix}9\\4\\-8\end{bmatrix}$ Linearly independent?

$$\left[\begin{array}{ccc|c}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & -6 & -8 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & 0 & 4 & 0\end{array}\right] \Rightarrow$$

unique solution, so the only soln to $A\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.

If you have n vectors & they span an n-dimensional space, then they're independent.

#33 - 38

33 T (thm 7)

34 T (thm 9)

35 F ($\bar{v}_1 = \bar{0}$)

36 F

37 T

38 T