MAT225 Section Summary: 2.1
Matrix Operations
Summary

As we saw when rotating vectors using matrix/vector products, we are going to want to multiply matrices upon matrices. We may also want to add matrices together. So we need to develop an "arithmetic" for matrices.

Sums of matrices $A+B$ only make sense when the matrices are exactly the same size (have the same number of rows/columns). Operations are carried out in the obvious ways, element-wise. If you multiply a matrix A by a scalar r, all the entries of the matrix are multiplied by r.

this: $[\ ]+[\ ]$

but not $[\ ]+[\ ]$

Exercise 3, p. 116

$$A = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix} \quad \text{Compute}$$

$$3I_2 - A = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix}$$

I_2: Identity 2×2

$$(3I_2)A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}_{2\times 2} \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix}_{2\times 2} = \begin{bmatrix} 12 & -3 \\ 15 & -6 \end{bmatrix}_{2\times 2}$$

$A\vec{x}=\vec{b} \Rightarrow A_{m\times n}$ then $\vec{x}$ is $n\times 1$

Products AB only makes sense when the number of columns of A matches the number of rows of B. One obvious consequence of this is that matrix multiplication is not commutative: in general, $AB \neq BA$. Multiplication is quite a new experience! In reality, multiplication is simply composition of linear transformations.

Exercise 8, p. 116

$$\underset{3\times ?}{B}\cdot\underset{?\times 4}{C} = [\quad]_{3\times 4}$$

Don't know what "?" is!

$$B_{3\times 4}\cdot C_{4\times 3} = [\quad]_{3\times 3}$$

$$B_{3\times 8}\cdot C_{8\times 3} = [\quad]_{3\times 3}$$

Given $A_{m \times n}$ and $B_{n \times p}$. The ij^{th} element of AB is the result of computing the inner product of the i^{th} row of A and the j^{th} column of B. In terms of individual components,

j^{th} column of B

$$(AB)_{ij} = \sum_{k=0}^{n} a_{ik} b_{kj}$$

i^{th} row of A

"Slam together" (inner product)

The matrix AB will be an m x p matrix.

Exercise 6, p. 116

$$\underset{3\times 2}{\begin{bmatrix} 4 & -2 \\ -3 & 0 \\ 3 & 5 \end{bmatrix}} \underset{2\times 2}{\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}} = \underset{3\times 2}{\begin{bmatrix} 0 & 14 \\ -3 & -9 \\ 13 & 4 \end{bmatrix}}$$

$\langle 4,-2\rangle \cdot \langle 1,2\rangle$

$\langle 4,-2\rangle \cdot \langle 3,-1\rangle$

Theorem 2, p. 113: Let $A_{m \times n}$, and let B and C have sizes for which the indicated sums and products are defined. Then

$A(BC) = (AB)C$	associative law of multiplication
$A(B+C) = AB + AC$	left distributive law
$(B+C)A = BA + CA$	right distributive law
$r(AB) = (rA)B = A(rB)$	for any scalar r
$I_m A = A = A I_n$	identity for matrix multiplication

The final property in this list indicates that some matrices commute: the identity matrix I_m commutes with any m x m matrix. But some don't:

Exercise 9, p. 116

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$$

$$AB = \underset{2\times 2}{\begin{bmatrix} 23 & 5k-10 \\ -9 & 15+k \end{bmatrix}}$$

$AB = BA \Rightarrow$

$$BA = \underset{2\times 2}{\begin{bmatrix} 23 & 15 \\ 6-3k & 15+k \end{bmatrix}}$$

$$\left.\begin{aligned} 5k-10 &= 15 \\ -3k+6 &= -9 \end{aligned}\right\}$$ Two equations, one unknown

$\Rightarrow \boxed{k = 5}$

Another curious property of matrix multiplication is that the usual cancellation laws do not hold: $AB = AC$ does not imply that $B = C$.

Exercise 10, p. 116

Furthermore, $AB = 0$ does not imply that either A or B is the zero matrix.

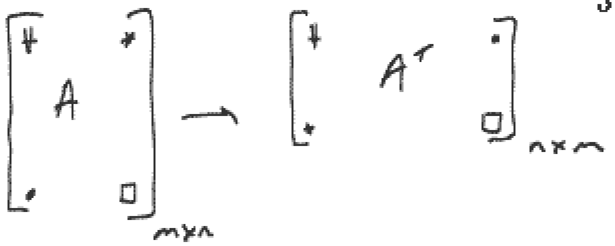
Exercise 12, p. 116

$$A = \begin{bmatrix} 2 & -4 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

$$\text{with } A \cdot B = 0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Definition: transpose. Given an m x n matrix A whose elements are a_{ij}. The **transpose** of A is the n x m matrix denoted A^T, whose entries are a_{ji}. Then

$$(A^T)^T = A$$
$$(A+B)^T = A^T + B^T$$
$$r(A^T) = rA^T$$
$$(AB)^T = B^T A^T$$



Exercise 27, p. 117.

$$\bar{u} = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\bar{u}^T \cdot \bar{v} = \begin{bmatrix} -2 & 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2a+3b-4c \end{bmatrix}_{1\times 1}$$

$$\bar{v}^T \bar{u} = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -2a+3b-4c \end{bmatrix} = \bar{u}^T \bar{v}$$

$$(\bar{v}^T \bar{u})^T = \bar{u}^T (\bar{v}^T)^T = \bar{u}^T \bar{v}$$

Exercise 33, p. 117.

$$\bar{u}\bar{v}^T = \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} -2a & -2b & -2c \\ 3a & 3b & 3c \\ -4a & -4b & -4c \end{bmatrix}$$

$$\bar{v}\bar{u}^T = \begin{bmatrix} -2a & 3a & -4a \\ -2b & 3b & -4b \\ -2c & 3c & -4c \end{bmatrix}$$

$\neq$ in general

The transpose may indicate to you why we've occasionally switched focus between rows and columns of a matrix: one minute you're studying $A\mathbf{x}$, the next minute you're studying $A^T\mathbf{x}$!