MAT225 Section Summary: 2.2

The Inverse of a Matrix

Summary

The inverse of a matrix is analogous to the multiplicative reciprical: we want to solve $A\mathbf{x} = \mathbf{b}$, and so we'd like to say that [redacted] but we don't know how to say that with matrices! Let's find out.... $\bar{x} = \bar{b}/A$

First of all, this concept only applies when matrices are **square**: so only n x n matrices could possibly be invertible.

Definition: inverse An n x n matrix A is **invertible** if theres exists an n x n matrix C (the **inverse** of A) such that

$$CA = I = AC$$

The inverse C is denoted A^{-1}, and is unique. A square matrix for which the inverse fails to exist is called **singular**.

A simple formula exists for the inverse of a two-by-two matrix: if A is given by

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Otherwise, A is singular. The quantity $ad - bc$ is called the **determinant** of A: $det(A) = ad - bc$.

~~#1, p. 126 (check!)~~

#17 §2.1 ✓

Theorem 5: if A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for each $\mathbf{b}$: $\mathbf{x} = A^{-1}\mathbf{b}$.

#5, p. 126 (check!)

Theorem 6:

1. If A is invertible, then $(A^{-1})^{-1} = A$.

   (Check #17, § 2.1

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \qquad A^{-1} = \frac{1}{1}\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = A \quad \checkmark$$

#1 p 124 $A = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix}$ $\quad A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ -5 & 8 \end{bmatrix}$ $\quad AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad (A^{-1})^{-1} = \begin{bmatrix} 2 & -3 \\ -5/2 & 4 \end{bmatrix}^{-1} = \frac{1}{\frac{1}{2}}\begin{bmatrix} 4 & 3 \\ 5/2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 5 & 4 \end{bmatrix} = A$

2. If A and B are n x n invertible matrices, then so is AB, and the inverse of AB is the product of the inverses, in the reverse order:

$$(AB)^{-1} = B^{-1}A^{-1}$$

   More generally, the inverse of a product of any number of invertible matrices is the product of the inverses in reverse order.

$$A \cdot A^{-1} = I \qquad \left(\tfrac{1}{2}A\right) \cdot 2\,A^{-1} \overset{?}{=} I \qquad \left(\tfrac{1}{2}A\right)^{-1} = 2A^{-1}$$

#15, p. 126. A, B, C invertible, $n \times n$

Show that ABC is also invertible, by producing D / $D(ABC) = (ABC)D = I$

$D = (ABC)^{-1} = C^{-1}B^{-1}A^{-1}$ matrix multiplication is associative

$D(ABC) = C^{-1}B^{-1}A^{-1}(ABC) = C^{-1}B^{-1}\underbrace{(A^{-1}A)}_{=I}BC$

$= C^{-1}(B^{-1}B)C = C^{-1}C = I$ ✓

3. If A is invertible, then so is A^T, and the inverse of A^T is the transpose of A^{-1}:

$$(A^T)^{-1} = (A^{-1})^T$$

Definition: an **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix. Each elementary matrix is invertible.

If an elementary row operation is performed on an m x n matrix A, the resulting matrix can be written as EA, where the m x m matrix E is created by performing the same row operation on I_m.

#28, p. 127

$$\underbrace{E_p \cdots E_3 E_2 E_1}A \sim I$$

In fact $= I$

Therefore, $(E_p \cdots E_1)$ is the inverse of A.

$$A = \begin{bmatrix} \bar{a}_{1\cdot} \\ \bar{a}_{2\cdot} \\ \bar{a}_{3\cdot} \end{bmatrix} \sim \begin{bmatrix} \bar{a}_{1\cdot} \\ \bar{a}_{2\cdot} \\ \bar{a}_{3\cdot} - 4\bar{a}_{1\cdot} \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{a}_{1\cdot} \\ \bar{a}_{2\cdot} \\ \bar{a}_{3\cdot} \end{bmatrix} = \begin{bmatrix} \bar{a}_{1\cdot} \\ \bar{a}_{2\cdot} \\ \bar{a}_{3\cdot} - 4\bar{a}_{1\cdot} \end{bmatrix}$$

Theorem 7: n x n matrix A is invertible if and only if A is row equivalent

So row reduce

$$[A \,\vdots\, I] \sim [I \,\vdots\, A^{-1}]$$

$E_p \cdot \ldots \cdot E_2 \cdot E_1 \cdot \underline{I} = E_p \cdot \ldots \cdot E_2 \cdot E = A^{-1}$

to I_n. The elementary row operations that transform A into I_n simultaneously transforms I_n into A^{-1}.

Theorem 7 suggests a method for finding A^{-1}: row reduce the augmented matrix $[AI_n]$. If A is row equivalent to I_n, then $[AI_n]$ is row equivalent to $[I_nA^{-1}]$.

#1, p. 126

#18, p. 126

#19, p. 126

#21, p. 126

Explain why the columns of A are linearly independent when A is invertible.

If A is invertible, then the only soln. of $A\bar{x} = \bar{0}$ is $\bar{x} = 0$ ($A^{-1}A\bar{x} = \bar{0} \Rightarrow \bar{x} = \bar{0}$) hence the columns are lin. indep.

Note : A^{-1} is generally not calculated: we don't need to know its entries to solve $A\mathbf{x} = \mathbf{b}$ (similar to the notion that we don't need to row reduce to reduced row echelon form to solve: we can stop with a triangular matrix).

① Once you get to

whatever!

$$\begin{bmatrix} 1 & - & - \\ 0 & 1 & - \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \end{bmatrix} = \begin{bmatrix} \bar{\alpha} \end{bmatrix}$$

you can just use backsubstitution to solve for $\bar{x}$ from $\bar{\alpha}$.