MAT225 Section Summary: 2.2 The Inverse of a Matrix Summary

The inverse of a matrix is analogous to the multiplicative reciprical: we want to solve $A\mathbf{x} = \mathbf{b}$, and so we'd like to say that but we don't know how to say that with matrices! Let's find out.... $\mathbf{x} = \mathbf{b} / \mathbf{a}$

First of all, this concept only applies when matrices are square: so only $n \times n$ matrices could possibly be invertible.

Definition: inverse An $n \times n$ matrix A is invertible if theres exists an $n \times n$ matrix C (the inverse of A) such that

$$CA = I = AC$$

The inverse C is denoted A^{-1} , and is unique. A square matrix for which the inverse fails to exist is called singular.

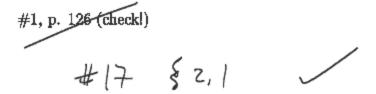
A simple formula exists for the inverse of a two-by-two matrix: if A is given by

$$A = \left[\begin{array}{cc} a & b \\ c & d \end{array} \right]$$

then, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Otherwise, A is singular. The quantity ad - bc is called the **determinant** of A: det(A) = ad - bc.



Theorem 5: if A is invertible, then $A\mathbf{x} = \mathbf{b}$ has a unique solution for each b: $\mathbf{x} = A^{-1}\mathbf{b}$.

#5, p. 126 (check!)

Theorem 6:

1. If A is invertible, then $(A^{-1})^{-1} = A$.

(Check #17,
$$\frac{3}{2}$$
]
$$A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \qquad A^{-1} = \frac{1}{1} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

$$(A^{-1})^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} = A$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 4^{-1} \\ 5 & 4 \end{bmatrix} \qquad A^{-1} = \frac{1}{2} \begin{bmatrix} 4^{-1} \\ 5 & 8 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 2 & -3 \\ 5 & 4 \end{bmatrix} = \frac{1}{2}$$

2. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses, in the reverse order:

$$(AB)^{-1} = B^{-1}A^{-1} \qquad \qquad = \begin{bmatrix} 7 & \downarrow \\ 5 & 4 \end{bmatrix} = \Delta$$

More generally, the inverse of a product of any number of invertible matrices is the product of the inverses in reverse order.

$$A \cdot A^{-1} = I \left(\frac{1}{2}A\right) \cdot 2 A^{-1} = I \left(\frac{1}{2}A\right)^{-1} = 2A^{-1}$$

3. If A is invertible, then so is A^T , and the inverse of A^T is the transpose of A^{-1} :

$$(A^T)^{-1} = (A^{-1})^T$$

Definition: an elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix. Each elementary matrix is invertible.

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m .

#28, p. 127

$$E_{p} = E_{3}E_{z}E_{z}A \sim I$$

$$I_{n} \text{ fact} = I$$

Theorem 7: $n \times n$ matrix A is invertible if and only if A is row equivalent

$$E_{\rho} \cdot ... \cdot E_{z} \cdot E_{1} \cdot \underline{T} = E_{\rho} \cdot ... \cdot E_{z} \cdot E = A^{-1}$$

to I_n . The elementary row operations that transform A into I_n simultaneously transforms I_n into A^{-1} .

Theorem 7 suggests a method for finding A^{-1} : row reduce the augmented matrix $[AI_n]$. If A is row equivalent to I_n , then $[AI_n]$ is row equivalent to $[I_nA^{-1}]$.

#1, p. 126

#18, p. 126

#19, p. 126

#21, p. 126

Explain why the colonors of A are linedy independent when A is invertible.

If A: invertible, then the only sola.

of $A\bar{x} = \bar{0}$ is $\bar{x} = 0$ (A'A $\bar{x} = \bar{0} = 7$)

hence the colonors are line indep.

Note: A^{-1} is generally not calculated: we don't need to know its entries to solve $A\mathbf{x} = \mathbf{b}$ (similar to the notion that we don't need to row reduce to reduced row echelon form to solve: we can stop with a triangular matrix).

Once you get to

you can just use backsubstitution to solve for x from Z.