MAT225 Section Summary: 2.3

Characterizations of Invertible Matrices

Summary

Theorem 8: The Invertible Matrix Theorem

Let A be a square n x n matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

1. A is invertible.
2. A is row equivalent to the identity matrix.
3. A has n pivot positions.
4. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
5. The columns of A form a linearly independent set.
6. The linear transformation $\mathbf{x} \to A\mathbf{x}$ is one-to-one.
7. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
8. The columns of A span $\mathbb{R}^n$.
9. The linear transformation $\mathbf{x} \to A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
10. There is an n x n matrix C such that $CA = I$.
11. There is an n x n matrix D such that $AD = I$.
12. A^T is invertible.

As the author says, "the power of the Invertible Matrix Theorem lies in the connections it provides between so many important concepts...."

#5, p. 132

$$A = \begin{bmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{bmatrix} \quad 1 \sim$$

not enough pivots

linear dependence in columns

#11, p. 132

#15, p. 132

Two identical columns $\Rightarrow$ linear dependence

No!

#17, p. 133

A invertible $\Rightarrow$ columns of A^{-1} linearly independent

$\Downarrow$

A^{-1} invertible $\nearrow$

True

#18, p. 133

$C_{6\times 6}$; 2 $C\bar{x} = \bar{v}$ consistent for every $\bar{v}$ in R^6.

Is it possible that $\exists$ more than one solution for $\bar{v}$

time comes to invert, and for other calculations. The **condition number** of a matrix measures how poorly conditioned a matrix is. The identity matrix has a condition number of 1, and is perfectly well-conditioned. The larger the condition number is, the closer a matrix is to singular (the condition number is infinite for a singular matrix). For a 2 x 2 matrix, the closer the determinant is to zero, the larger the condition number.

#42, p. 134

#28 If AB is invertible, so is B.
(we're to assume square A & B)

$$AB = \begin{bmatrix} A\bar{b}_1 & A\bar{b}_2 & \cdots & A\bar{b}_n \end{bmatrix}$$

So $(AB)^{-1}$ exists

Since AB is invertible, its columns form a linearly independent set of vectors.

From last time, if the columns of B are linearly dependent, then so are the columns of AB.

The columns of AB are not dependent.

Therefore the columns of B are not dependent; They're independent, so B is invertible by IMT.

#26 Explain why the columns of A^2 span $\mathbb{R}^n$

whenever the columns of A are linearly independent.

A^{-1} exists by IMT.

$A^2 = A \cdot A$.

Product of invertible matrices is invertible (thm 6), so A^2 is invertible. By IMT, the columns of A^2 span $\mathbb{R}^n$.