MAT225 Section Summary: 2.5

Matrix Factorization

Summary

Matrix factorization of matrix A means to break a matrix into a product. It is carried out for (at least) two different reasons:

- because it's advantageous to think of A as a series of successive linear transformations, or
- to bring out some structure in the matrix A.

We're focusing on the LU decomposition, which is one strategy for solving linear equations $A\mathbf{x} = \mathbf{b}$. Rather than compute A^{-1} (when possible) and multiply $\mathbf{b}$ by it, it's more advantageous to simply factor $A=LU$, where L is lower-triangular and U is upper-triangular.

This process is better conditioned numerically, and may have other advantages: for example, if A is sparse (has lots of zeros), the LU decomposition may also have many zeros, but the inverse A^{-1} tends to be ``full'' (that is, of non-zero elements).

In order to solve $A\mathbf{x} = \mathbf{b}$, we proceed as follows:

$$A\mathrm{x} = LU\mathrm{x} = \mathrm{b},$$

Two equations for 1 - bad deal?

so we can solve this in two steps: L & b known; find $\bar{y}$

$L(U\bar{x}) = b$

$$L\mathrm{y} = \mathrm{b}$$

They're both easy to solve by back substitution

and

then

$$U\mathrm{x} = \mathrm{y}$$

U & $\bar{y}$ are known, solve for $\bar{x}$.

It doesn't seem that we've made much progress, since we've replaced one equation by two, until we notice that it's easy to solve both of the new equations since they're triangular. If we need to solve many equations of the form $A\mathbf{x} = \mathbf{b}$, with fixed A, then it often makes sense to first factor into LU.

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If A is square $[A|I] \sim [I|A^{-1}]$

The algorithm for finding the LU factorization is simple:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I (and such that the ``diagonal entries'' (those with equal indices, a_{ii}) of L are one - remember, L is not necessarily square).

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LU decomposition stops the row reduction process short, at

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$[A|I] \sim [U|?]$

How do we get U?

by a series of row operations,

$$\underbrace{(E_p \cdots E_2 E_1)}A = U$$

If invertible, then

$$A = (E_p \cdots E_1)^{-1} U$$

$$= \underbrace{E_1^{-1} \cdots E_p^{-1}}_{L} U$$

If L is lower triangular

(it will be if we don't do any pivoting ...)

$$L = (E_p \cdots E_1)^{-1}$$

$$= E_1^{-1} \cdots E_p^{-1}$$

$$E_1 L = E_2^{-1} \cdots E_p^{-1}$$

$$E_2 E_1 L = E_3^{-1} \cdots E_p^{-1}$$

$$E_p \cdots E_1 L = I$$

$$\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} L \end{bmatrix} \begin{bmatrix} U \end{bmatrix}$$

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$$A = \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

scalar multiple of 1st column to give 1 on the diagonal

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\quad L \qquad\qquad\qquad U$$

$$L(\underbrace{U\bar{x}}_{=\bar{y}}) = \bar{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$$

$y_1 = 2$

$y_2 = -2$

$y_3 = 2$

$$\begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1/4 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix} \begin{bmatrix} 1/4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \quad ✓$$

$$\sim \begin{bmatrix} 4 & 3 & -5 \\ 0 & -2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

Scale it so there's a 1 on the diagonal

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & \boxed{1} \end{bmatrix}$$

do it one more time (scale 2 to get 1!)

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$$A = \begin{bmatrix} -5 & 3 & 4 \\ 10 & -8 & -9 \\ 15 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 10 & 14 \end{bmatrix}$$

scalar multiples

$$\sim \begin{bmatrix} -5 & 3 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & 9 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & \boxed{-5} & 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 2 & -6 & 4 \\ -4 & 5 & -7 \\ 3 & 5 & -1 \\ -4 & 4 & -8 \\ 8 & -3 & 9 \end{bmatrix} \sim \begin{bmatrix} 2 & -6 & 6 \\ 0 & -7 & 5 \\ 0 & 14 & -10 \\ 0 & -14 & 10 \\ 0 & 21 & -15 \end{bmatrix} \sim \underbrace{\left[\begin{array}{ccc} 2 & -6 & 6 \\ 0 & -7 & 5 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]}_{U}$$

$$L = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 3/2 & -2 & 1 & 0 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 4 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 3/2 & -2 \\ -3 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 2 & -6 & 4 \\ 0 & -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3/2 \\ -3 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & -6 & 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ -3 \end{bmatrix} \begin{bmatrix} 0 & -7 & 5 \end{bmatrix}$$