MAT225 Section Summary: 4.1
Vector Spaces and Subspaces
Summary

One of the interesting things about vector spaces is that they're applicable in many unusual ways. We're used to thinking of vectors as arrows, with magnitude and direction. But we're going to expand the idea of vectors to functions of real variables, discrete time series, matrices (for goodness sake!), and other mathematical objects.

The starting point is the list of properties found in section 1.3, p. 32. Those, in addition to two "closure" properties, are the properties described below:

vector space: A vector space is a nonempty set V of objects (called vectors) on which are defined two operations, called "addition" and "multiplication by scalars", and subject to the rules below:

real numbers

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u}+\mathbf{v}$, is in V (V is closed under addition).
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ (addition is commutative)
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$ (addition is associative)
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ (additive identity).
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$ (additive inverses).
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V (V is closed under multiplication).
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$ (multiplication is distributive over vector addition).
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$ (multiplication is distributive over scalar addition).
9. $c(d\mathbf{u})=(cd)\mathbf{u}$ (multiplication is associative).
10. $1\mathbf{u}=\mathbf{u}$

For each $\mathbf{u}$ in V and scalar c,

$$0\mathbf{u} = \mathbf{0}$$
$$c\mathbf{0} = \mathbf{0}$$
$$-\mathbf{u} = (-1)\mathbf{u}$$

#1 $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \geq 0 \right\}$

a. $\bar{u} + \bar{v} = \begin{bmatrix} x_u \\ y_u \end{bmatrix} + \begin{bmatrix} x_v \\ y_v \end{bmatrix} = \begin{bmatrix} x_u + x_v \\ y_u + y_v \end{bmatrix}$

$x_u + x_v \geq 0$

$y_u + y_v \geq 0$ so

$\bar{u} + \bar{v} \in V$

b. $\bar{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$c = -1$

$-\bar{v} \notin V$

Examples (or not!): p. 223, #1, 7, 8

<u>**subspace**: A subspace of a vector space</u> V is a subset H of V that has three properties:

1. The zero vector of V is in H.
2. H is closed under vector addition. That is, for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.
3. H is closed under multiplication by scalars. That is, for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

The trivial subspace contains only the zero vector, and clearly satisfies the three properties above.

Example: p. 224, #21

Another important example is given in the following theorem:

Theorem 1: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V: the subspace spanned by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Given any subspace H of V, a **spanning set** for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in H such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

#7 $V = \{ p(x) \mid p(x) = ax^3 + bx^2 + cx + d$ and $a, b, c, d \in$ Integers $\}$

is not a vector space: 2

Consider "$\bar{v} = 3x^3 - 7$"

$\pi \bar{v} = 3\pi x^3 - 7\pi \notin V$.

#21 p224

$V = \{ M_{2\times 2} \mid 2\times 2 \text{ matrices w/ real components} \}$

& matrix addition & usual scalar multiplication is a vector space.

$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, c \text{ real} \right\}$; is H a subspace (using the same operations!) ?

1) $\bar{0} \in H$ $\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ - take } a = b = d = 0 \right)$

2) Closed under addition?

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} e & f \\ 0 & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ 0 & d+h \end{bmatrix} \in H \checkmark$$

3) Closed under scalar mult. :

$$\alpha \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ 0 & \alpha d \end{bmatrix} \in H \checkmark$$

So yes, H forms a subspace.

Alternatively H is the spanning set of 3 vectors :

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so H is a subspace.

#17 $$\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$