MAT225 Section Summary: 4.2

Null spaces, column spaces, and linear transformations

Summary

$\bar{x} \in \mathbb{R}^n$

The solution set of the homogeneous equation $A_{m \times n}\mathbf{x} = \mathbf{0}$ forms a subspace of $\mathbf{R}^n$, as one can see easily:

$\bar{0} \in \mathbb{R}^m$

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1. the zero vector is in the solution set (the trivial solution);
2. Consider two vectors in the solution set, $\mathbf{u}$ and $\mathbf{v}$: then $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$, so the solution set is closed under addition.
3. Consider a vectors in the solution set, $\mathbf{u}$ and an arbitrary constant c: then $A(c\mathbf{u}) = cA\mathrm{u} = \mathbf{0}$, so the solution set is closed under scalar multiplication.

Null space of an m x n matrix A: the null space of an m x n matrix A, denoted Nul A, is the solution set of the homogeneous equation $A\mathbf{x} = \mathbf{b}$. It is the set of all $\mathbf{x} \in \mathbf{R}^n$ that are mapped to the zero vector of $\mathbb{R}^m$ by the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Theorem 2: The null space of an m x n matrix A is a subspace of $\mathbf{R}^n$.

Example: #3, p. 234.

$$\underset{2\times 4}{A} = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \qquad \text{Nul } A \subseteq \mathbb{R}^4 \qquad A\bar{x} = \bar{0}.$$

$$\begin{aligned} x_1 &= 7x_3 - 6x_4 \\ x_2 &= -4x_3 + 2x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned} \qquad \bar{x} = \begin{bmatrix} 7x_3 - 6x_4 \\ -4x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Notice that the number of vectors in the spanning set for Nul A equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Column space: Another subspace associated with the matrix A is the column space, Col A, defined as the span of the columns of A: Col $A =$ Span $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. As a span, it is clearly a subspace (Theorem 3).

Col $A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$ in $\mathbb{R}^n\}$, which says that Col A is the range of the transformation $\mathbf{x} \longrightarrow A\mathbf{x}$.

Example: #16, p. 234

$$\left\{ \underbrace{\begin{bmatrix} b - c \\ 2b + c + d \\ 5c - 4d \\ d \end{bmatrix}}_{\bar{v}} : b, c, d \text{ real} \right\}$$

$$\bar{v} = b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

$$A_{4\times ?} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

The null space lives in the row space ($\mathbb{R}^n$) of the matrix A, and the column space lives in the column space ($\mathbb{R}^m$) of A.

Example: #22, p. 235

Linear Transformation: A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

2. T(cu) = cT(u)

The **kernel** (or **null space**) of T is the set of $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

Example: #30, p. 235 $\quad T: V \to W$

① $T(\bar{0}) = \bar{0} \Rightarrow \bar{0} \in$ Range of T

② Given $T(\bar{x})$ + $T(\bar{w})$; $T(\bar{x}) + T(\bar{w}) = T(\bar{x}+\bar{w})$

So $T(\bar{x}) + T(\bar{w})$ is in the Range, image of $\bar{x}+\bar{w}$;

③ $cT(\bar{x}) = T(c\bar{x})$, so $cT(\bar{x})$ is in the range of T, image of $c\bar{x}$. So $T(V)$ is a subspace of W.

Examples of linear transformations include matrix transformations, as well as differentiation in the vector space of differentiable functions defined on an interval (a, b).

Example: #33, p. 235 $\quad M_{2\times 2}$, real coefficients.

$T: M_{2\times2} \to M_{2\times2}$ by $T(A) = A + A^T$

a) $T(A_1+A_2) = A_1 + A_2 + (A_1+A_2)^T = A_1 + A_2 + A_1^T + A_2^T$

$= A_1 + A_1^T + A_2 + A_2^T = T(A_1) + T(A_2)$ ✓

$T(cA_1) = cA_1 + (cA_1)^T = cA_1 + cA_1^T = cT(A_1)$ ✓

continued below....

Column space: is it a subspace?

$C = \{ \bar{b} \in R^m \mid \exists\, \bar{x} \mid^3 A\bar{x} = b \}$

① $\bar{0} \in C$, because $A\bar{0} = \bar{0}$

② Given $\bar{b}_1$ + $\bar{b}_2$ in C.

$A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{b}_1 + \bar{b}_2$

So $\bar{b}_1 + \bar{b}_2$ is in C, the image of $\bar{x}_1 + \bar{x}_2$.

③ $c\bar{b}_1$ is in C, the image of $c\bar{x}_1$:

$$A(c\bar{x}_1) = cA\bar{x}_1 = c\bar{b}_1 \checkmark$$

#33 continued....

b. Let $B \in M_{2\times 2}$ / $B^T = B$. Find $A \in M_{2\times 2}$ / $B = T(A)$.

$$T(A) = A + A^T \underbrace{= B}_{\text{demand this}}$$

Let $A = \frac{1}{2}B$; $T(A) = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B$
$= B \checkmark$

c) Show that $Range(T)$ is $\{B \in M_{2\times 2} / B^T = B\}$

$T(A) = A + A^T$

$[T(A)]^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A$
$= A + A^T = T(A)$.

So every $T(A)$ is equal to its own transpose $\checkmark$

d. Describe the kernel of T

$$T(A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = A + A^T$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ also satisfies $T(A) = \bar{0}$

So the kernel is $c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, a one dimensional subspace of $M_{2\times 2}$.