MAT225 Section Summary: 4.4
Coordinate Systems
Summary

A basis gives us a way of writing each vector $\mathbf{v}$ in a vector space in a unique way, as a linear combination of the basis vectors. The coefficients of the basis vectors can be considered the coordinates of $\mathbf{v}$ in a coordinate system determined by the basis vectors.

Theorem 7: the Unique Representation Theorem

Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$$

If not, $\bar{x} = d_1\bar{b}_1 + \ldots + d_n\bar{b}_n$; but then $\bar{x} - \bar{x} = (c_1 - d_1)\bar{b}_1 + \ldots + (c_n - d_n)\bar{b}_n = \bar{0}$; but since the $\bar{b}_i$ are independent, the coefficients are zero: $c_i = d_i$.

Coordinates: Suppose $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V, and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** B are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$.

$$[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the **coordinate vector of** x **(relative to** B **)**, or the **B-coordinate vector of** x. The mapping

$$\mathbf{x} \mapsto [\mathbf{x}]_B$$

is the **coordinate mapping (determined by** B).

Example: #1, p. 253

$$B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\} \qquad [x]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\bar{x} = 5\begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}}_{P_B}\begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Let

$$P_B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \cdots \ \mathbf{b}_n]$$

Then

$$\mathbf{x} = P_B[\mathbf{x}]_B$$

is the link between the standard basis representation of $\mathbf{x}$ (on the left) and the representation of $\mathbf{x}$ in the basis B.

Example: #5, p. 254

$$B = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\} \qquad \bar{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} \quad \checkmark$$

$$= \begin{bmatrix} 1 & 2 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -2 \\ -3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -5 \end{bmatrix} \qquad c_2 = -5 \quad c_1 = 8 \quad [x]_B = \begin{bmatrix} 8 \\ -5 \end{bmatrix}$$

Example: #14, p. 254

$$B = \{ \underset{\bar{b}_1}{1 - t^2}, \underset{\bar{b}_2}{t - t^2}, \underset{\bar{b}_3}{2 - 2t + t^2} \} \qquad p(t) = 3 + t - 6t^2$$

$$\bar{p} = c_1 \bar{b}_1 + c_2 \bar{b}_2 + c_3 \bar{b}_3$$

$$\begin{bmatrix} 1 & 0 & 2 & \vdots & 3 \\ 0 & 1 & -2 & \vdots & 1 \\ -1 & -1 & 1 & \vdots & -6 \end{bmatrix} \sim \ldots \Rightarrow \bar{c} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} = [\bar{p}]_B$$

Theorem 8: Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_B$ is a one-to-one linear transformation from V onto $\mathbf{R}^n$.

This is an example of an *isomorphism* ("same form") from V onto W. These spaces are essentially indistinguishable.

Example: #23, p. 254 Vector space V, basis $\{\bar{b}_1, \ldots, \bar{b}_n\}$

$x : V \to \mathbb{R}^n$

Consider two elements of $\mathbb{R}^n$ $[\bar{u}]_B$ + $[\bar{w}]_B$; show that $\bar{u} = \bar{w}$ if $[\bar{u}]_B = [\bar{w}]_B$. Let $[\bar{u}]_B = [w]_B$. Then

$\bar{u} = c_1\bar{b}_1 + c_2 b_2 + \ldots + c_n b_n$

and

$\bar{w} = c_1\bar{b}_1 + c_2\bar{b}_2 + \ldots + c_n\bar{b}_n$; therefore $\bar{u} = \bar{w}$!

Example: #24, p. 254

Consider the polynomials

$p_1(t) = 1 + t$

$p_2(t) = 1 - t$

$p_3(t) = 2$

$B_1 = \{\bar{p}_1, p_2\}$ $\quad$ $\bar{p}_3 = \underbrace{1\bar{p}_1 + 1\cdot\bar{p}_2}$

$$[p_3]_{B_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$B_2 = \{\bar{p}_1, \bar{p}_3\}$

$$[p_3]_{B_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\bar{p}_3 = 0\cdot\bar{p}_1 + 1\cdot\bar{p}_3$