MAT225 Section Summary: 4.5
The Dimension of a Vector Space
Summary

Theorem 9: If a vector space V has a basis $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 10: If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

dimension of a vector space: If V is spanned by a finite set, then V is **finite-dimensional**, and the dimension V (dim V) is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

Example: #2, p. 260

$$\begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix} = s\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Two-dimensional subspace of $\mathbb{R}^3$

Theorem 11 Let H be a subspace of a finite-dimensional vector space V. Any linearly indendent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Example: #11, p. 261

$$\text{rref}\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix}$$

1

find the # of pivot columns.

Theorem 12 (the Basis Theorem): Let V be a p-dimensional vector space, $p \leq 1$. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.

Example: #22, p. 261

$\{1,\ 1-t,\ 2-4t+t^2,\ 6-18t+9t^2-t^3\}$

$$\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Has 4 pivots, spans a 4-dimensional space, in a 4-dimensional space! I.e. it's all of $\mathbb{P}_3$.

Let A be an m x n matrix. Then the dimension of Nul A is the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$, and the dimension of Col A is the number of pivot columns in A.

n = basic + free

basic variables

Example: #14, p. 261

3 basic $\Rightarrow$ 3 is dim of Col(A)

3 free $\Rightarrow$ 3 is dim Nul(A)

$A = \begin{bmatrix} & & \end{bmatrix}_{m \times n}$

$\mathrm{Nul}(A) \subseteq \mathbb{R}^n$ (dimension: # of free)

$\mathrm{Col}(A) \subseteq \mathbb{R}^m$ (dimension: # of basic)

Example: #27, 28, p. 262

A takes $\mathbb{R}^n$ into $\mathbb{R}^m$

#27: basis for $\mathbb{P}$: $\{1, t, t^2, t^3, \ldots\}$

#28: $C(\mathbb{R})$ contains $\mathbb{P}$, which is infinite dimensional; hence it too must be infinite dimensional.