MAT225 Section Summary: 4.6

Rank

Summary

Rank: The rank of a matrix is the dimension of the column space of A. That is, it is equal to the number of independent vectors among the columns of the matrix.

row space: the row space of a matrix A is the span of the rows of A.

Theorem 13: If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the non-zero rows of B form a basis for the row spaces of A and B.

Theorem 14 (The Rank Theorem): The dimensions of the column space and the row space of an m x n matrix A are equal (the rank of A). The rank satisfies the relation

$$\text{rank } A + \dim \text{Nul } A = n$$

You may be wondering why the Nul space popped up here: the point is that all these spaces are fundamentally connected.

Example: #2, p. 269

$$\text{rank } A = 3$$

$$A \sim \begin{bmatrix} 1 & -3 & 0 & 5 & -7 \\ 0 & 0 & 2 & -3 & 8 \\ 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \dim \text{null } A = 5 - 3 = 2$$

$$B_C = \left\{ \begin{bmatrix} 1 \\ -2 \\ -3 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 9 \\ -10 \\ -3 \\ 0 \end{bmatrix} \right\} \qquad B_R = \left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ 8 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 5 \end{bmatrix} \right\} \qquad B_N = \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The Invertible Matrix Theorem (continued): Let A be an n x n matrix. Then the following statements are each equivalent to the statement that A is an invertible matrix:

$$x_1 + -3x_1 + 0x_3 + 5x_4 + -7x_5 = 0$$

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$$2x_3 + -3x_4 + 8x_5 = 0$$

$$5x_5 = 0 \Rightarrow x_5 = 0$$

$$x_1 = 3x_2 - 5x_4$$

$$x_2$$

$$x_3 = \tfrac{3}{2}x_4$$

$$x_4$$

$$x_5 = 0$$

$$x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$$

A square!

- The columns of A form a basis of $\mathbb{R}^n$.
- Col $A = \mathbb{R}^n$
- dim Col A = n
- rank A = n
- Nul $A = \{\mathbf{0}\}$
- dim Nul $A = 0$

Examples: #5,8-11, p. 269

Example: #16, p. 269

Example: #18, p. 270