MAT225 Section Summary: 5.1

Eigenvalues and Eigenvectors
Summary

We're considering the transformation $A_{n \times n} : \mathbf{x} \mapsto A\mathbf{x}$. Eigenvectors provide the ideal basis for $\mathbf{R}^n$ when considering this transformation. Their images under the transformation are simple scalings.

Eigenstuff: An **eigenvector** of $A_{n \times n}$ is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. The scalar λ is called the **eigenvalue** of A corresponding to $\mathbf{x}$. There may be several eigenvectors corresponding to a given λ.

The idea is that an eigenvector is simply scaled by the transformation, so the actions of a transformation are easily understood for eigenvectors. If we could write a vector as a linear combination of eigenvectors, then it would be easy to calculate its image: if there are n eigenvectors $\mathbf{v}_i$, with n eigenvalues λ_i, then if

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$$

then

$$A\mathbf{u} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \ldots + c_n\lambda_n\mathbf{v}_n$$

Nice, no?

¡A HREF="http://klebanov.homeip.net/tildarooniepavel/fb//java/la_applets/Eigen/"¿Eigenv
applet¡/A¿

If λ is an eigenvalue of matrix A corresponding to eigenvector $\mathbf{v}$, then

$$A\mathbf{v} = \lambda\mathbf{v}$$

This means the

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0}$$

which is equivalent to

$$(A - \lambda I)\mathbf{v} = \mathbf{0}$$

If $\det(A-\lambda I) = 0$, then there's a non-trivial null-space — so v is in λ!

So $\mathbf{v}$ is in the null space of $A - \lambda I$. If the null space is trivial, then $\mathbf{v}$ is the zero vector, and λ is not an eigenvalue. Alternatively, all vectors in the null space are eigenvectors corresponding to the eigenvalue λ.

As for determining the eigenvectors and eigenvalues, there is some cases in which this is extremely easy:

The eigenvalues of a diagonal matrix are the entries on its diagonal. More generally,

Theorem 1: The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an n x n matrix A, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ is linearly independent.

n distinct eigenvalues of an nxn matrix $\Rightarrow$ n eigen-vectors form a basis.

The eigenvectors and difference equations portion of this section can be illustrated with the example of the Fibonacci numbers transformation: recall that the Fibonacci numbers are those obtained by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

and $F_0 = 1$ and $F_1 = 1$.

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}_n = \mathbf{x}_{n+1}$$

$$A\bar{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} ; \quad A^2\bar{x}_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

where

$$x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A^3\bar{x}_0 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The eigenvalues of this matrix are approximately $\gamma = \frac{1+\sqrt{5}}{2} \approx 1.618033988749895$ and -0.618033988749894. γ is the so-called "golden mean", which is a nearly sacred number in nature, well approximated by the ratio of consecutive Fibonacci numbers.

An eigenvector corresponding to the golden mean (normalized to have a norm of 1) is approximately

$$\begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix}$$

so that

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix} = \gamma \begin{bmatrix} 0.5257311121191337 \\ 0.8506508083520401 \end{bmatrix}$$

Example: full matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

Find eigenvalues: $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

Set $\det(A - \lambda I) = 0$

$$(1-\lambda)(1-\lambda) - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - (-1))(\lambda - 3) = 0$$

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3 \qquad (\text{distinct!})$$

Now we need corresponding eigenvectors:

$\lambda_1 = -1$: $A - (-1) I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

Spot an eigenvector! $\bar{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$: $A - 3I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$ $\bar{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

#2 $A - (-2) I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$

Columns are linearly dependent $\Rightarrow$ non-trivial null space. $\operatorname{Det}(A - (-2)I) = 9 - 9 = 0$.

$\bar{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ is $\qquad (A - (-2)I)\,\bar{v}_1 = \bar{0}$

<u>an</u> eigenvector

#9 $A = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}$ $\lambda_1 = 1,\ \lambda_2 = 5$

$(A - \lambda_1 I)\bar{v}_1 = \bar{0}$ $A - \lambda_1 I = \begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$

$\bar{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$(A - \lambda_2 I)\bar{v}_2 = \bar{0}$ $A - \lambda_2 I = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$

$\bar{v}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

So $\{\bar{v}_1, \bar{v}_2\}$ forms a basis for $\mathbb{R}^2$, since they correspond to distinct eigenvalues.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 0 \\ 2 & 1-\lambda \end{bmatrix}$$

Demand $\det(A - \lambda I) = 0$ for eigenvalues

$$(5-\lambda)(1-\lambda) = 0$$

Back to #2:

$$A - \lambda I = \begin{bmatrix} 7-\lambda & 3 \\ 3 & -1-\lambda \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$(7-\lambda)(-1-\lambda) - 9 = \lambda^2 - 6\lambda - 16$$
$$= (\lambda+2)(\lambda-8)$$
$$\lambda = -2 \qquad \lambda = 8$$