MAT225 Section Summary: 5.2

The Characteristic Equation

Summary

The Invertible Matrix Theorem (continued): Let A be an n x n matrix. Then A is an invertible matrix if and only if:

- The number 0 is not an eigenvalue of A.
- The determinant of A is *not* zero.

A scalar λ is an eigenvalue of $A_{n \times n}$ if and only if λ satisfies the **characteristic equation**:

$$\det(A - \lambda I) = 0$$

This is a polynomial of degree n in λ, called the **characteristic polynomial.** If a value of λ has a multiplicity of p as a root of the characteristic polynomial, then we say that the eigenvalue λ is said to have multiplicity p as an eigenvalue.

Example: Find the eigenvalues of the "Fibonacci" matrix

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow$$

$$(-\lambda)(1-\lambda) - 1 = 0$$

$$\boxed{\lambda^2 - \lambda - 1} = 0$$

characteristic polynomial

$$\lambda = \frac{-(-1) \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$\boxed{r = \frac{1+\sqrt{5}}{2}}$$

Example: #5, p. 317

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = \lambda^2 - 6\lambda + 9$$

$$= (\lambda - 3)^2$$

"eigenvalues" − 3! Repeated ... possible trouble

Example: #15, p. 318

Eigenvalues = $\{4, 3, 1\}$, 3 repeated.

similarity: If $A_{n \times n}$ and $B_{n \times n}$ satisfy the relation

$$A = PBP^{-1}$$

(where P is clearly invertible), then A and B are said to be **similar**.

Note: we are now giving a meaning to the word "similar" that is inconsistent with row-equivalence: two matrices A and B are **row-equivalent** if there is an invertible matrix E such that $A = EB$.

Theorem 4: If n x n matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Example: #24, p. 318 (this is straight out of the proof of Theorem 4).

A, B similar $\Rightarrow \exists P \;/\; A = PBP^{-1}$

Show that $\det(A) = \det(B)$.

$$\det(A) = \det(PBP^{-1}) = \det P \cdot \det B \cdot \det P^{-1}$$

$$= \underbrace{\det P \cdot \det P^{-1}} \cdot \det B$$

$$= \det(P \cdot P^{-1}) \cdot \det B = \det(I) \det B = \det B$$

Example: #25, p. 318

$$A = \begin{bmatrix} .6 & .3 \\ .4 & .7 \end{bmatrix} \qquad \bar{v}_1 = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} \quad 2 \qquad \bar{x}_0 = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$$

$$A\bar{v}_1 = \begin{bmatrix} 18/70 + 12/70 \\ 12/70 + 28/70 \end{bmatrix} = \begin{bmatrix} 30/70 \\ 40/70 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} = \bar{v}_1, \text{ so } \lambda_1 = 1$$

$$A - \lambda I = \begin{bmatrix} .6-\lambda & .3 \\ .4 & .7-\lambda \end{bmatrix} \qquad \det(A - \lambda I) = (.6-\lambda)(.7-\lambda) - .12$$

$$= \lambda^2 - 1.3\lambda + .42 - .12$$
$$= \lambda^2 - 1.3\lambda + .3$$
$$= (\lambda - 1)(\lambda - .3)$$

$\lambda_2 = .3$

$$A - .3I = \begin{bmatrix} .3 & .3 \\ .4 & .4 \end{bmatrix} \qquad \bar{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \text{ for example}$$

$\mathcal{B} = \{\bar{v}_1, \bar{v}_2\}$ is a basis for $\mathbb{R}^2$, since the 2 eigenvectors correspond to distinct eigenvalues.

b. Verify that $\bar{x}_0 = \bar{v}_1 + c\,\bar{v}_2$

$$\checkmark \qquad \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} 3/7 \\ 4/7 \end{bmatrix} + \begin{bmatrix} .5/7 \\ -.5/7 \end{bmatrix} = \bar{v}_1 + 5/70\,\bar{v}_2$$

c)

$$\bar{x}_k = A^k \bar{x}_0 = A^k(\bar{v}_1 + 5/70\,\bar{v}_2)$$
$$= \bar{v}_1 + 5/70\,A^k \bar{v}_2$$
$$= \bar{v}_1 + 5/70\,(.3)^k\,\bar{v}_2$$

$$\lim_{k \to \infty} \bar{x}_k = \bar{v}_1$$