MAT225 Section Summary: 5.3

Diagonalization

Summary

diagonalizable: A square matrix A is diagonalizable if A is similar to a diagonal matrix. That is, if $A = PDP^{-1}$ for some diagonal matrix D.

The Diagonalization Theorem: $A_{n \times n}$ is diagonalizable if and only if A has n linearly independent eigenvectors. Moreover, $A = PDP^{-1}$ (where D is diagonal) if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are the eigenvalues.

$AP = PD$

Example: #2, p. 325

Rewrite the equation $A = PDP^{-1}$ in the form $AP = PD$ to understand what is going on! This is just the eigenvalue equation in partitioned form:

$$A\underbrace{[\mathbf{v}_1\ \mathbf{v}_2\ \dots\ \mathbf{v}_n]}_{\text{columns of } P} = \underbrace{[\lambda_1\mathbf{v}_1\ \lambda_2\mathbf{v}_2\ \dots\ \lambda_n\mathbf{v}_n]}_{\text{scaled columns of } P}$$

; e.g. $A\vec{v}_1 = \lambda_1\vec{v}_1$

Theorem 6: An n x n matrix with n distinct eigenvalues is diagonalizable.

Example: #10, p. 326

Theorem 7: Let A be an n x n matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \dots, \lambda_p$.

1. For $1 \le k \le p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

2. The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals n.

3. If A is diagonalizable, and B_k is a basis for the eigenspace corresponding to λ_k, then the collection of the bases $B_1, \dots, B_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Example: #33, p. 326

#2 p 325

$A = PDP^{-1}$

$A^k = PD^kP^{-1}$

$$A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

$$A^k = \underbrace{\begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}}_{\text{eigenvectors of } A,\ A^k} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^k \end{bmatrix}}_{\text{eigenvalues of } A^k} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}$$

#10 $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ $\qquad A - \lambda I = \begin{bmatrix} 2-\lambda & 3 \\ 4 & 1-\lambda \end{bmatrix}$

① Find eigenvalues (D)

$\det A = (\lambda - 5)(\lambda + 2)$

$\lambda_1 = 5$, $\lambda_2 = -2$ $\qquad D = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

② Find corresponding eigenvectors (P)

$A - \lambda_1 I = \begin{bmatrix} -3 & 3 \\ 4 & -4 \end{bmatrix}$ $\qquad \bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda - \lambda_2 I = \begin{bmatrix} 4 & 3 \\ 4 & 3 \end{bmatrix}$ $\qquad \bar{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$P = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$

③ Calculate P^{-1}, + write $A = PDP^{-1}$

$$A = \begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4/7 & 3/7 \\ -1/7 & 1/7 \end{bmatrix}$$