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Mathematica Maple

# Condition Numbers

**Text Reference: Section 2.3, p. 131**

The purpose of this set of exercises is to show how a condition number of a matrix $A$ may be defined, and how its value affects the accuracy of solutions to systems of equations $A\mathbf{x} = \mathbf{b}$.

Consider the following equation $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3+\epsilon & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 26+\epsilon \end{bmatrix}$$

Here $\epsilon$ is a number which may be changed.

**Questions**:

1. Confirm that $x_1 = 1$, $x_2 = 2$, $x_3 = 3$ is a solution to this system no matter the value of $\epsilon$.
2. Show that if $\epsilon \neq 0$, then $A$ is invertible, while if $\epsilon = 0$, then $A$ is not invertible.

Thus when $\epsilon$ is near $0$, then $A$ is "almost" not invertible, and this causes computers and calculators to sometimes have problems in calculating the solution to $A\mathbf{x} = \mathbf{b}$.

**Question**:

3. Use the row reduction capabilities of your technology to find the solution to the above system of equations with $\epsilon = 10^{-5}$, $\epsilon = 10^{-8}$ and $\epsilon = 10^{-11}$. You may need to ask your technology to report more digits than it usually does in order to see what is happening. How close are your answers to the true solution $x_1 = 1$, $x_2 = 2$, $x_3 = 3$; that is, how many digits of accuracy do the computed solutions have?

Apparently, the accuracy of the technology depends on how close a matrix is to not being invertible. Some notation and theory will be introduced to analyze this situation. Let $\mathbf{r}$ be the result returned by the technology in solving the system, let $\mathbf{x}$ be the true solution to the system, and let $\Delta\mathbf{x} = \mathbf{r} - \mathbf{x}$. Thus $\Delta\mathbf{x}$ measures the discrepency between the true solution $\mathbf{x}$ and the solution that the technology gives. The difference between the values $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{r}$ is also necessary: let $\Delta\mathbf{b} = A\mathbf{r} - \mathbf{b} = A(\mathbf{x} + \Delta\mathbf{x}) - \mathbf{b} = A\Delta\mathbf{x}$.

**Question**:

4. Calculate $\Delta\mathbf{x}$ and $\Delta\mathbf{b}$ for your results in Question 3 when

   a) $\epsilon = 10^{-5}$

   b) $\epsilon = 10^{-8}$

   c) $\epsilon = 10^{-11}$

The sizes of the vectors $\Delta \mathbf{x}$ and $\Delta \mathbf{b}$ are important, as are their sizes relative to the vectors $\mathbf{x}$ and $\mathbf{b}$. In order to do this a notion of the size of a vector is needed. There are many possible ways to define this notion of size, which is called a **norm**. One possible definition is given here.

**Definition**: The **norm** of a vector is the largest of the absolute values of the elements in the vector; that is for $\mathbf{x} = (x_1, x_2, \ldots x_n)$,

$$||\mathbf{x}|| = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

**Example**: If $\mathbf{x} = (1, -2, 0)$, then $||\mathbf{x}|| = \max\{|1|, |-2|, |0|\} = 2$.

**Questions**:

5. Find $||\mathbf{x}||$ for the following vectors.

   a) $\mathbf{x} = (-5, 7, 2, -4)$
   b) $\mathbf{x} = (4, -8, 3, 1)$
   c) $\mathbf{x} = (7, -9, 1, 0)$

6. Calculate $\frac{||\Delta \mathbf{x}||}{||\mathbf{x}||}$ and $\frac{||\Delta \mathbf{b}||}{||\mathbf{b}||}$ for your results in Question 3 when

   a) $\epsilon = 10^{-5}$
   b) $\epsilon = 10^{-8}$
   c) $\epsilon = 10^{-11}$

The size, or norm, of a matrix $A$ may also be defined.

**Definition**: The **norm** of a matrix is

$$||A|| = \max\left\{\frac{||A\mathbf{v}||}{||\mathbf{v}||} : \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq \mathbf{0}\right\}.$$

Thus the norm of a matrix is the maximum amount by which a matrix may stretch any nonzero vector $\mathbf{x}$:

$$||A\mathbf{x}|| = \frac{||A\mathbf{x}||}{||\mathbf{x}||} \cdot ||\mathbf{x}|| \leq \max\left\{\frac{||A\mathbf{v}||}{||\mathbf{v}||} : \mathbf{v} \in \mathbb{R}^n, \mathbf{v} \neq \mathbf{0}\right\} \cdot ||\mathbf{x}|| = ||A|| \cdot ||\mathbf{x}|| \qquad (1)$$

It turns out that different vector norms give us different matrix norms. The *definition* of the norm is difficult to use in practice with specific matrices. However, it can be shown that when the vector norm defined above is used, the associated matrix norm is found by adding up the absolute values of the elements in each row of the matrix, and then selecting the largest such row sum. That is, for an $m \times n$ matrix $A$,

$$||A|| = \max\left\{\sum_{j=1}^{n} |a_{ij}| : 1 \leq i \leq m\right\}$$

**Example:** If

$$\begin{bmatrix} 1 & 2 & 3 \\ -3 & 1 & 4 \\ 0 & -2 & 4 \end{bmatrix},$$

then the row sums are $1+2+3=6$, $3+1+4=8$, and $0+2+4=6$, thus $||A||=8$.

**Question**:

7. Find the norms of the following matrices.

a) $\begin{bmatrix} 4 & -4 & 3 \\ 7 & -4 & -5 \\ 7 & -7 & 6 \end{bmatrix}$

b) $\begin{bmatrix} 2 & -4 & -9 \\ -3 & 8 & 3 \\ 1 & 7 & -5 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 & -4 \\ -6 & 5 & -7 \\ -3 & 3 & -3 \end{bmatrix}$

How does the norm of a matrix apply to the analysis of the system $A\mathbf{x} = \mathbf{b}$? If $A\mathbf{x} = \mathbf{b}$, then by Equation 1 above,

$$||\mathbf{b}|| \leq ||A|| \cdot ||\mathbf{x}||. \tag{2}$$

If $A$ has an inverse $A^{-1}$, then $\Delta\mathbf{x} = A^{-1}(\Delta\mathbf{b})$. Again by Equation 1,

$$||\Delta\mathbf{x}|| \leq ||A^{-1}|| \cdot ||\Delta\mathbf{b}||. \tag{3}$$

The sizes of $\Delta\mathbf{x}$ and $\Delta\mathbf{b}$ relative to the sizes of vectors $\mathbf{x}$ and $\mathbf{b}$ are of interest now. The ratios $\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}$ and $\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$ are called the **relative errors** of, respectively, $\Delta\mathbf{x}$ and $\Delta\mathbf{b}$. For the user of technology to be confident in the solution technology gives to $A\mathbf{x} = \mathbf{b}$, $\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}$ should be small; the difference between the true solution and the computed solution should be small relative to size of the true solution.

**Question**:

8. Combine Equations 2 and 3 to show that

$$\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||} \leq ||A|| \cdot ||A^{-1}|| \cdot \frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$$

Thus the relative error in the result $\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}$ cannot be more than $||A|| \cdot ||A^{-1}||$ times the relative error in $\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$. The number $||A|| \cdot ||A^{-1}||$ is crucial to the analysis.

**Definition**: Given a vector norm on $\mathbb{R}^n$ and an invertible $n \times n$ matrix $A$, the **condition number** of $A$ is

$$\text{cond}(A) = ||A|| \cdot ||A^{-1}||$$

**Question**:

9. Find the condition number of

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3+\epsilon & 4 & 5 \end{bmatrix}$$

when

a) $\epsilon = 10^{-5}$

b) $\epsilon = 10^{-8}$

c) $\epsilon = 10^{-11}$

What happens to the condition number as $\epsilon$ approches $0$? If you have a computer algebra system, find $A^{-1}$ in symbolic form and use it to compute the condition number of $A$ as a function of $\epsilon$. Then discuss what happens to the condition number of $A$ as $\epsilon$ approaches zero.

Even though the condition number gives valuable information about the system $A\mathbf{x} = \mathbf{b}$, the knowledge of the system is still incomplete. Since $\frac{||\Delta\mathbf{b}||}{||\mathbf{b}||}$ depends on the true solution $\mathbf{x}$ which in practice is unknown, an absolute bound on the size of $\frac{||\Delta\mathbf{x}||}{||\mathbf{x}||}$ cannot be given. However, there is a useful rule of thumb which can let us know approximately how accurate the solution to $A\mathbf{x} = \mathbf{b}$ is. It is given in the text on page 133.

**Rule of Thumb**: If the entries in $A$ and $\mathbf{b}$ are accurate to $m$ significant digits, and if the condition number of $A$ is approximately $10^k$ (with $k$ a positive integer), then the computed solution of $A\mathbf{x} = \mathbf{b}$ should usually be accurate to at least $m - k$ significant digits.

**Questions**:

10. Consider the following equation studied in Reference 1.

$$\begin{bmatrix} 888,445 & 887,112 \\ 887,112 & 885,781 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   a) Confirm that $x_1 = 885,781$ and $x_2 = -887,112$ is a solution to this equation.
   b) Use your technology to row reduce an appropriate matrix and record its solution. How accurate is it?
   c) Compute the condition number of $A$.
   d) Find out how many significant digits your technology stores accurately and suppose that the entries in $A$ and $\mathbf{b}$ are that accurate. To how many significant digits is the solution in b) expected to be accurate?

11. For each of the following $4 \times 4$ matrices $A$, find the condition number. Construct a random vector $\mathbf{x}$ in $\mathbb{R}^4$, and let $\mathbf{b} = A\mathbf{x}$. Solve the system $A\mathbf{x} = \mathbf{b}$ using your technology. In this case you will know both the true solution $\mathbf{x}$ and the computed solution $\mathbf{r}$. How many digits of accuracy does your technology's solution have? How many digits of accuracy does the condition number cause you to expect?

   a) $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 10 & 5 & 3 & 2 \\ 18 & 40 & 45 & 45 \\ 20 & 15 & 12 & 10 \end{bmatrix}$

   b) $\begin{bmatrix} 9 & 1 & 19 & 7 \\ 2 & -4 & -9 & 0 \\ 1 & -2 & -5 & 0 \\ 3 & 2 & 26 & 3 \end{bmatrix}$

   c) $\begin{bmatrix} 31 & 11 & 21 & -9 \\ 54 & 22 & 38 & -9 \\ -2 & -4 & -2 & -4 \\ 3 & 1 & 2 & -1 \end{bmatrix}$

**Notes**:

1. Many calculators and software programs have the capability of finding a condition number of a matrix; in fact, many programs will warn you when you are attempting to solve a system whose coefficient matrix has a large condition number.

2. The condition number that most software programs calculate and the condition number discussed on page 478 of the text are not the same as the condition number in this exercise set. That condition number is generated by a **different vector norm**. However, the interpretation of the condition number is consistent: the larger the condition number, the closer the matrix is to not being invertible.

**Reference:**

1. Nievergett, Yves. "Numerical Linear Algebra on the HP-28 or How to Lie With Supercalculators." *American Mathematical Monthly*, June-July 1991, pp. 539-543.