#9 §2.4

$$\lim_{n\to\infty} \frac{|b_{n+1} - a_{n+1}|}{|b_n - a_n|^{\alpha}} = \frac{1}{2} \; \lambda$$

$$\lim_{n\to\infty}$$

$$= \lim_{n\to\infty} \frac{\frac{b-a}{2^{n+1}} - 0}{\frac{b-a}{2^n} - 0} = \frac{1}{2} = 1/2$$

Suppose

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|} = \lambda$$

Then

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} \qquad \alpha \neq 1$$

$$\begin{cases} \alpha < 1 & = 0 \\ \alpha > 1 & = \infty \end{cases}$$

$$= \lim_{n\to\infty} \frac{|p_{n+1}-p|}{|p_n - p|^1} \cdot \frac{1}{|p_n - p|^{\alpha-1}}$$

$$p_n = \frac{1}{2^n}$$

$$\frac{\left|\frac{1}{2^{n+1}} - 0\right|}{\left|\frac{1}{2^n} - 0\right|^{\alpha}} = \frac{\frac{1}{2^{n+1}}}{\left(\frac{1}{2^n}\right)^{\alpha}} \quad \left(= \lambda \text{ in the limit} \right)$$

$$= \frac{2^{\alpha n}}{2^{n+1}} = 2^{\boxed{\alpha n - n}+1} \Rightarrow \alpha n - n = 0$$

#6 §1, 3

a) $\lim_{n\to\infty} \sin\frac{1}{n} = 0$

$$\sin x = x - \frac{x^3}{3!} + O(x^5)$$

$$= x + O(x^3)$$

$$\sin\frac{1}{n} = \frac{1}{n} + O\left(\frac{1}{n^3}\right)$$

$$\frac{\sin\frac{1}{n}}{1/n} = 1 + O\left(\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} = 1$$

So certainly $\left| \sin \frac{1}{n} - 0 \right| \le 2 \frac{1}{n}$

for large n.

Set $h = \frac{1}{n}$.

Consider $\sin h = 0 + \frac{\cos \xi}{1!} h$

$$|\sin h - 0| = |(\cos \xi) h|$$

$$\leq 1 h$$

$$\left|\sin \frac{1}{n} - 0\right| \leq 1 \cdot \frac{1}{n}$$

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d. $\lim_{n \to \infty} [\ln(n+1) - \ln(n)] = 0$

$$\lim_{n \to \infty} \left[\ln \frac{n+1}{n}\right] = \lim_{n \to \infty} \ln\left(1 + \frac{1}{n}\right)$$

$$f(x) = \ln(1+x) = \ln 1 + 1x + O(x^2)$$

Expand in Taylor series about $x=0$

$$[\ln(1+x)]' = \frac{1}{1+x}$$

$$\ln\left(1+\frac{1}{n}\right) = 0 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$$\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1 + O\left(\frac{1}{n}\right)$$

$$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = 1$$

$$\left| \ln\left(1+\frac{1}{n}\right) - 0 \right| \le 2\,\frac{1}{n}$$

7c. $$\lim_{h\to 0} \frac{\sinh - h\cosh}{h} = 0$$

$$\sinh = h - \frac{h^3}{3!} + O(h^5)$$

$$\cosh = 1 - \frac{h^2}{2!} + O(h^4)$$

$$\underline{\sinh - h\cosh} = \left(h - \frac{h^3}{3!} + O(h^5)\right)$$

$$-h\left(1-\frac{h^2}{2!}+O(h^4)\right)$$

$$= -\frac{h^3}{3!}+\frac{h^3}{2!}+O(h^5)$$

$$\sinh - h\cosh = \tfrac{1}{3}h^3 + O(h^5)$$

$$\lim_{h\to 0} \frac{\sinh - h\cosh}{h} = \lim_{h\to 0} \tfrac{1}{3}h^2 + O(h^4)$$

$$\left|\frac{\sinh - h\cosh}{h}\right| \le \tfrac{2}{3}|h^2|$$