MAT360 Section Summary: 1.3

Algorithms and Convergence

1. Summary

There are three separate issues discussed in this section:

- algorithms,
- convergence and growth of errors, and
- order of error (little o and big O).

An algorithm is a recipe for completing a task. As we've seen, algorithms giving the same answer from the purely mathematical standpoint may give radically different answers from a numerical perspective. So we want to make good choices when we create algorithms.

If an algorithm has the property that small changes in initial conditions produce small changes in the solution, then the algorithm is **stable**; otherwise it is **unstable**. Some algorithms are stable for a range of initial data, and they might be categorized as **conditionally stable**.

If errors introduced at the outset grow linearly, i.e. as

$$E_n \approx CnE_0$$

where C is independent of n. If, on the other hand, the errors grow exponentially,

$$E_n \approx C^n E_0 \qquad (C > 1)$$

If $C < 1$, errors are decreasing!

then we're probably going to be in trouble before we'd like!

2. Definitions

- **Definition 1.18** Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence which converges to zero, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If $\exists K > 0$ with

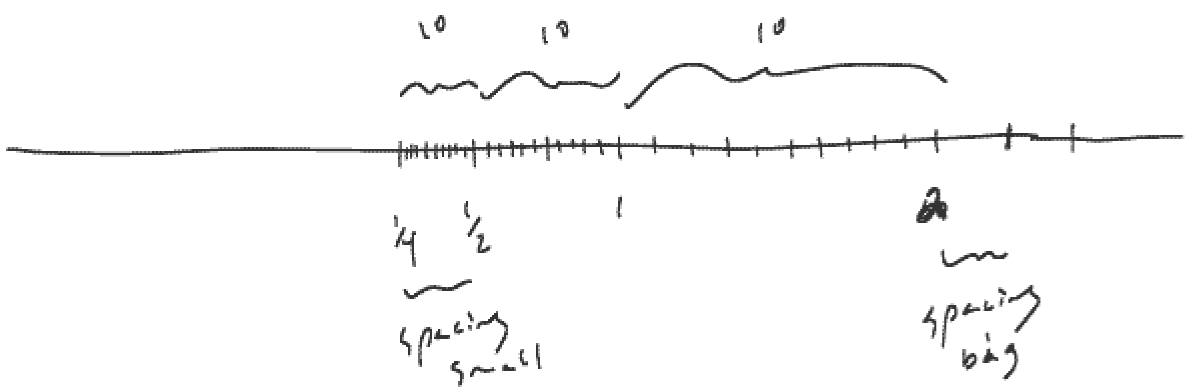
  $$|\alpha_n - \alpha| \leq K|\beta_n|$$

  for large n, then $\{\alpha_n\}_{n=1}^{\infty}$ converges to α with **rate of convergence** $O(\beta_n)$.

- **Definition 1.19**: Suppose that $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$. If $\exists K > 0$ /
$$|F(h) - L| \leq K|G(h)|,$$
for sufficiently small h, then $F(h) = L + O(G(h))$.

3. Properties/Tricks/Hints/Etc.

β_n is generally of the form of powers of $\frac{1}{n}$.



$$\boxed{\rho_n = \frac{1}{6}\cdot \rho_{n-1} \qquad \rho_0 = 1} \qquad \rho_n = \frac{1}{6}^n \quad \forall n \in [0,1,..]$$

$$\rho_n^* = .16\ \rho_{n-1}^* \qquad \rho_0^* = 1 \qquad \rho_n^* = (.16)^n$$ [two-digit chopped model for ρ]

$$|\rho_1 - \rho_1^*| = .00\bar{6} = \frac{2}{3}\times 10^{-2}$$

$$|\rho_n - \rho_n^*| = \left|\frac{1}{6}\rho_{n-1} - .16\,\rho_{n-1}^*\right| = \left|.16(\rho_{n-1} - \rho_{n-1}^*) + \frac{2}{3}\times 10^{-2}\,\rho_{n-1}\right| \geq$$

$$|\rho_n - \rho_n^*| \geq .16\,|\rho_{n-1} - \rho_{n-1}^*|$$

$$E_n \geq .16 \quad E_{n-1}$$

bounding below

$$\frac{1}{6} = .16 + \frac{2}{3}\times 10^{-2}$$

$$\frac{1}{6} p_{n-1} = .16 p_{n-1} + \frac{2}{3}\times 10^{-2} p_{n-1}$$

We're also interested in bounding above:

$$E_n \leq p_n$$

$$\left(|p_n - p_n^*| = \left|\frac{1}{6} p_{n-1} - .16 p_{n-1}^*\right| \leq \frac{1}{6} p_{n-1} = p_n \right)$$

We've got E_n bounded!

$$p_n \geq E_n \geq .16 E_{n-1}$$

(absolute error)

Relative error:

100% error!

$$1 \geq \frac{E_n}{p_n} \geq .16 \frac{E_{n-1}}{p_n}$$

If we'd rounded to two digits:

$$p_n^* = .17 p_{n-1}^*$$

& the conclusions would have been dramatically different:

The relative error would have have grown exponentially to ∞! Here are the details:

$$E_n = |p_n - p^*_{n-1}| = \left|\tfrac{1}{6}p_{n-1} - .17\,p^*_{n-1}\right| = \left|\underbrace{\tfrac{1}{6}p_{n-1} - \tfrac{1}{6}p^*_{n-1}}_{\text{negative}} - (.17-\tfrac{1}{6})p^*_{n-1}\right|$$

$$= \left|\tfrac{1}{6}(p^*_{n-1} - p_n) + (.17 - \tfrac{1}{6})p^*_{n-1}\right|$$

$$\geq \tfrac{1}{6}|p^*_{n-1} - p_n| = \tfrac{1}{6}|p_n - p^*_{n-1}| = \tfrac{1}{6}E_n$$

$$E_n = |p_n - p^*_{n-1}| \leq .17\,p^*_{n-1} = p^*_n \qquad \text{(being brutal!)}$$

Hence

$$p^*_n \geq \boxed{E_n \geq \tfrac{1}{6}E_{n-1}}$$

So the relative error satisfies

$$\frac{p^*_n}{p_n} \geq \frac{E_n}{p_n} \geq \frac{\tfrac{1}{6}E_{n-1}}{p_n}$$

How do we handle this inequality?

By solving the equality

$$E_n = \tfrac{1}{6}E_{n-1} \qquad E_1 = \tfrac{1}{3}\times 10^{-2}$$

$$E_n = \tfrac{1}{6}^{n-1}\cdot\tfrac{1}{3}\times 10^{-2}$$

& realizing that E_n is greater than this! So

$$\frac{p^*_n}{p_n} \geq \frac{E_n}{p_n} \geq \frac{\tfrac{1}{6}^{n-1}\cdot\tfrac{1}{3}\times 10^{-2}}{p_n} \qquad p_n = \tfrac{1}{6}^n$$

$$p^*_n = (.17)^n$$

$$\underbrace{(1.02)^n \geq \frac{E_n}{p_n}}\geq 2\times 10^{-2}$$

This gives exponential error as a possibility ...

and, as we discussed in class, the likely outcome.

#6c. $$\lim_{n\to\infty}\left(\sin\frac{1}{n}\right)^2 = 0$$

$$\sin x = x + O(x^3)$$

$$\sin\frac{1}{n} = \frac{1}{n} + O\left(\left(\frac{1}{n}\right)^3\right)$$

$$\left(\sin\frac{1}{n}\right)^2 = \left(\frac{1}{n} + O\left(\left(\frac{1}{n}\right)^3\right)\right)\left(\frac{1}{n} + O\left(\left(\frac{1}{n}\right)^3\right)\right)$$

$$= \frac{1}{n^2} + O\left(\frac{1}{n^4}\right)$$

$$\left(\left(\sin\frac{1}{n}\right)^2 - 0\right] = \frac{1}{n^2} + O\left(\frac{1}{n^4}\right)$$

7d $$\lim_{h\to 0}\frac{1-e^h}{h} = -1$$

$$e^x = 1 + \frac{1}{1!}x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{n=0}^{\infty}\frac{x^n}{n!} \qquad \boxed{0! = 1}$$

$$e^h = 1 + h + \frac{h^2}{2} + \dots = 1 + h + O(h^2)$$

$$\frac{1-e^h}{h} = \frac{1-[1+h+O(h^2)]}{h}$$

$$= -1 - \frac{O(h^2)}{h}$$

$$= -1 - O(h)$$

$$\frac{1-e^h}{h} - (-1) = -O(h)$$