MAT360 Section Summary: ~~1.4~~ 2.1
The Bisection Method

1. Summary

We're getting into root finding now: the problem is one that you're very familiar with: find x such that

$$f(x) = 0$$

We're going to start finding roots with the simplest, most obvious method: trap one! In order to trap one, we pinch it with the Intermediate Value Theorem. So we'll require continuity of f.

This is an iterative scheme, which means that we'll be doing the same thing over and over, until satisfied. Since it's iterative, we could implement it in several ways, including having the algorithm call itself. This is called **recursion**. It's the most fun, but possibly quite dangerous (since one could spiral down forever and ever if not careful).

The name "bisection" suggests the scheme: we're going to cut an interval in half each time. So we need an interval to start, and since we're planning on using the IVT to search for a root, we need one $[a, b]$ such that $f(a) * f(b) \leq 0$.

So the idea is this:

- Find a and b such that $f(a) * f(b) \leq 0$ (if 0, either a or b is a root);
- Choose a tolerance ϵ and a "tolerance scheme" (this is the error we're willing to accept in our "solution" c, related to any of several stopping criteria): for example
  - $|f(c)| < \epsilon$
  - $|c - r| < \epsilon$
  - $|(c - r)/r| < \epsilon$ (watch that $r \neq 0$)
- Compute $c = mean(a, b)$ (how could this be improved?);
- Compute $f(c)$ (if 0, exit: c is the root);

- If $f(a) * f(c)$ 0[1] then
$$b = c$$
else
$$a = c$$
- Check the error tolerance: for example, if you're using the criterion that the $|c - r| < \epsilon$, then compute $d = |b - a|$;
- If $d < 2 * \epsilon$ then the mean is within ϵ, so
  - exit with root $mean(a, b)$;

  else do it all again!
  - bisect(a,b,eps)

You might also put in a stopping criterion, so that if you've done a certain magic number of iterations, the best root to that point will be produced.

2. Definitions

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$$\text{signum}(x) = \begin{array}{rl} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{array}$$

3. Theorems/Formulas

Theorem 2.1: Suppose that $f \in C[a, b]$ and $f(a) * f(b)$ 0. The bisection method generates a sequence $\{r_n\}_{n=1}^{\infty}$ approximating a root r of f with
$$|r_n - r| \leq \frac{b-a}{2^n},$$
when $n \geq 1$.

[1] As mentioned in the text, it's better to multiply signum values, rather than the actual function values. This is because it's easy to do the sign of numbers – we just check that one bit out of 64, for example! And multiplying ones and negative ones is easy....

4. Properties/Tricks/Hints/Etc.

Notice the calculation of the mean of a and b as

$$a + \frac{b-a}{2}$$

This is better conditioned than

$$\frac{a+b}{2}$$

in some cases: for example, with two-digit numbers chopped, and $a = 98$ and $b = 99$, then $a + b = 190$; so $\frac{a+b}{2} = \cancel{100}$ 95! That's a funny average.... But

$$a + \frac{b-a}{2} = 98 + .5 = 98$$

(still bad, but at least we're still in the interval $[a, b]$!). Of course it could lead to problems if using recursion, or do not employ a stopping criterion based on a maximum # of iterations.

#16, §1.3

$$F_{n+2} = F_{n+1} + F_n$$

$$x_n = \frac{F_{n+1}}{F_n}$$ Assuming $\lim_{n\to\infty} x_n = x$ exists,

Show that $$x = \frac{1+\sqrt{5}}{2}$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n-1} = x$$

$$\frac{F_n}{F_{n-1}} = \frac{F_{n-1} + F_{n-2}}{F_{n-1}} = 1 + \frac{F_{n-2}}{F_{n-1}} = 1 + \frac{1}{\left(\frac{F_{n-1}}{F_{n-2}}\right)}$$

$$\boxed{x_{n-1} = 1 + \frac{1}{x_{n-2}}} \geq 1$$

$$\lim_{n\to\infty}\left[x_{n-1} = 1 + \frac{1}{x_{n-2}}\right] \text{"="} \; x = 1 + \frac{1}{x}$$

$$\lim_{n\to\infty}\left[x_{n-1} - \left(1 + \frac{1}{x_{n-2}}\right)\right] = \lim_{n\to\infty} 0 = 0$$

$$x - \left(1 + \frac{1}{x}\right) = 0$$

[Need to $x=0$ from consideration.]

Solve $\boxed{x^2 - x - 1 = 0}$

6d $\ln(n+1) - \ln n =$ small, & getting smaller as $n \to \infty$

$$= \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right)$$

Taylor Series (about $x=0$):

$$f(x) = \ln(1+x) = 0 + x$$

$$f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$$

$$f'(x) = \frac{1}{1+x} \qquad f'(0) = 1$$

$$\ln\left(1 + \frac{1}{n}\right) = 0 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

$$\ln\left(1 + \frac{1}{n}\right) - 0 = \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

" $\lim_{n \to \infty} \left[\ln\left(1+\frac{1}{n}\right) - 0 \right]$ is $O(1/n)$ "

$$\ln \frac{\left(1+\frac{1}{n}\right) - 0}{1/n} = 1 + \frac{O(1/n^2)}{1/n} = 1 + O(1/n)$$