**MAT360 Section Summary: 2.2**
Fixed-Point Iteration

## Summary

Suppose that you want to solve the equation

$$\cos(x) = x$$

The value of $x$ that satisfies this equation is called a **fixed point** for the function $g(x) = \cos(x)$, because it is a point such that $g(x) = x$ – the image is the same as the argument.

One way to go about finding the fixed point would be to rewrite the equation as

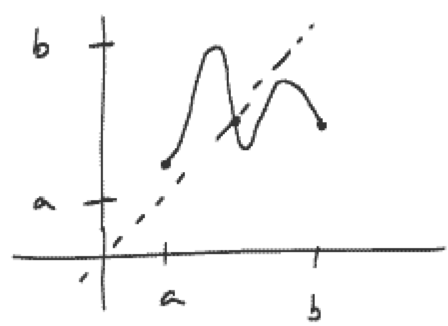
$$f(x) = \cos(x) - x = 0$$

and to use bisection to find a root of $f$ (in fact, the unique root, as one can see from the graph of $f$).

Fixed-point iteration is based on a couple of results from calculus: the IVT, and the MVT, as follows:

**Theorem 2.2**:

- If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then $g$ has a fixed point in $[a,b]$.
- If, in addition, $g'(x)$ exists on $(a,b)$ and a positive constant $k < 1$ exists with
$$|g'(x)| \le k, \quad \forall x \in (a,b),$$
then the fixed point in $[a,b]$ is unique.




The proofs are by

- the IVT, with $h(x) = g(x) - x$; and

If $g(a)=a$ or $g(b)=b$
then we've found our fixed point!

Otherwise,
$h(a)>0$ and $h(b)<0$ ; $h$ is continuous on $[a,b]$, hence by the IVT, $h$ has a root $p$.

So $h(p)=0 \iff g(p)-p=0 \iff \boxed{g(p)=p}$

– $p$ is a fixed point of $g$.

- the MVT, and contradiction.

Assume that there are two distinct fixed points, $p_1$ & $p_2$ — $g(p_1) = p_1$ & $g(p_2) = p_2$

$$|g(p_1) - g(p_2)| = |g'(c)(p_1 - p_2)|$$ by the MVT $c \in [p_1, p_2]$

$$|p_1 - p_2| = |g'(c)||p_1 - p_2|$$

$$\Rightarrow \quad |g'(c)| = 1$$

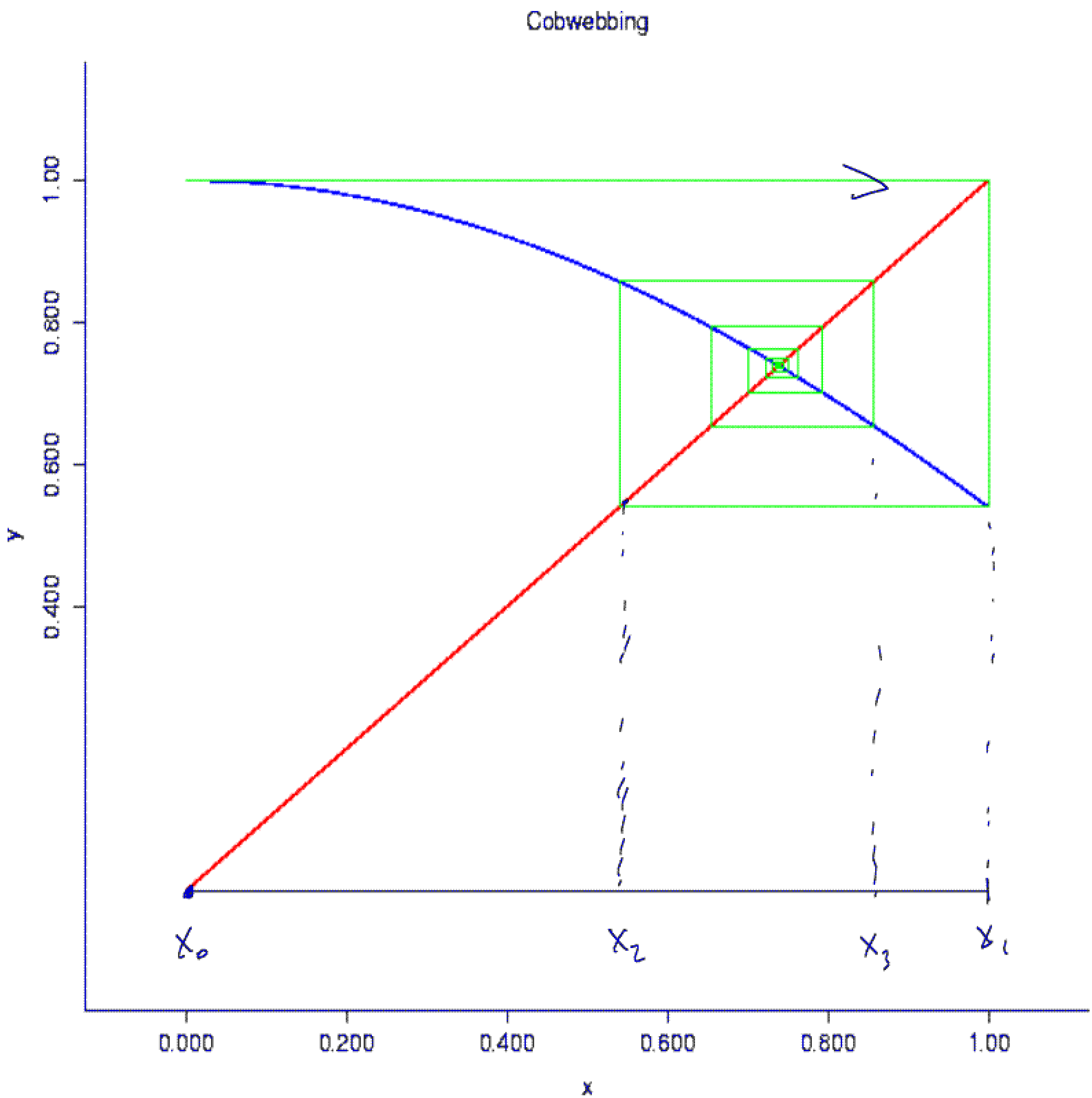
But by assumption, $|g'(x)| < 1$ for $(a,b)$; this is a contradiction of distinct fixed points. Hence the fixed point is unique.

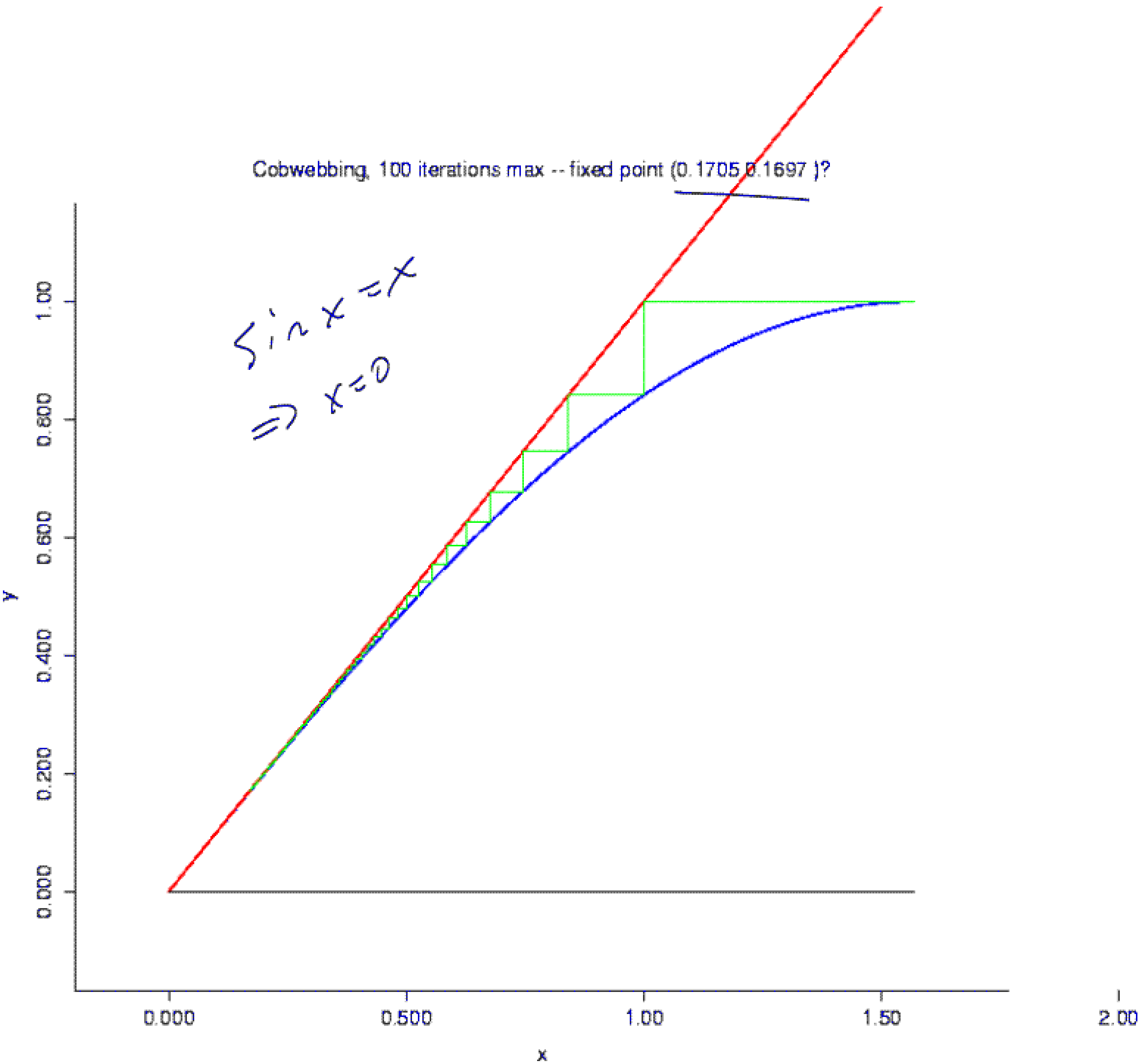
Actually the condition is sufficient but not necessary

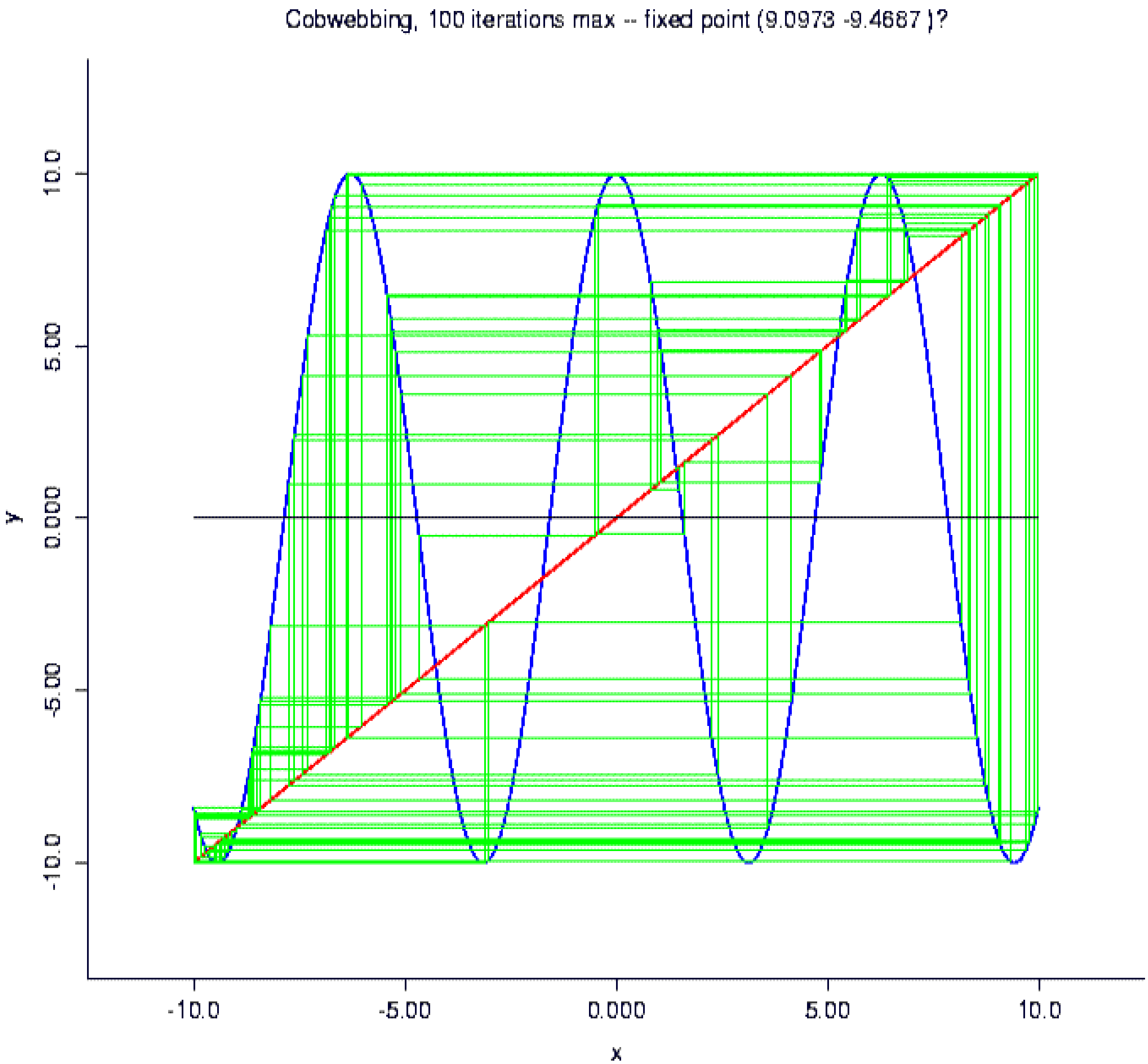
So we know that there's a fixed point on an interval $[a, b]$, and may even know that it's unique. What now?

Now we assume that, perhaps, if we start with a value $x_0$ that's close to the real fixed point $p$, that by simply computing $g(x_0)$ (which is $\approx x_0$) we'll actually get closer to $p$.

Let's look at the "cobweb diagram" of this situation.




Cobwebbing, 100 iterations max -- fixed point (0.1705 0.1697 )?
sin x = x
⇒ x = 0
y
x
0.000
0.200
0.400
0.600
0.800
1.00
0.000
0.500
1.00
1.50
2.00


Cobwebbing, 100 iterations max -- fixed point (9.0973 -9.4687 )?
10.0
5.00
0.000
-5.00
-10.0
y
-10.0
-5.00
0.000
5.00
10.0
x

Under what circumstances will that happen? In what circumstances would the same "cobwebbing" procedure fail?

Well, in some circumstances, it's guaranteed to work:

**Theorem 2.3: Fixed-Point Theorem** Let $g \in C[a,b]$ be such that $g(x) \in [a,b]$, for all $x$ in $[a,b]$. Suppose, in addition, that $g'$ exists on $(a,b)$, and that a constant $0 < k < 1$ exists with

$$|g'(x)| \leq k$$

for all $x \in (a,b)$. Then for any number $p_0 \in [a,b]$, the sequence

$$p_n = g(p_{n-1})$$

$n \geq 1$, converges to the unique fixed point $p$ in $[a,b]$.

**Proof**: MVT applied to $|p_n - p|$.

$$|p_n - p| = |g(p_{n-1}) - g(p)| = |g'(c)(p_{n-1} - p)| \leq k|p_{n-1} - p|$$

where $c$ is between $p_{n-1}$ & $p$

$$|p_n - p| \leq k^n |p_0 - p|$$

**Corollary 2.4**: If $g$ satisfies the hypotheses of Theorem 2.3, then bounds for the error involved in using $p_n$ to approximate $p$ are given by

$$|p_n - p| \leq k^n \max\{p_0 - a, b - p_0\}$$

and

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0|$$

for all $n \geq 1$.

$|p_m - p_n| =$

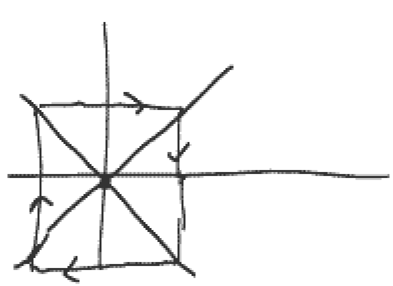
$|p_m - p_{m-1} + p_{m-1} - p_{m-2} + p_{m-2} + \dots p_n|$

**Proof**: by use of various inequalities.

There may be lots of ways to create a fixed-point function, and some of them are better than others.

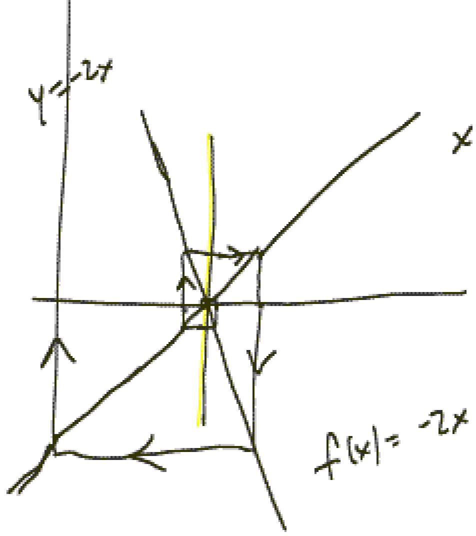
**Example**: consider exercises 1 and 2.





$-x = x$

"neutrally stable" — not getting any better, but not getting any worse!




spiralling out when |slope| > 1

#1

$$f(x) = x^4 + 2x^2 - x - 3$$

To solve: $f(x) = 0$

$$g_0(x) = x^4 + 2x^2 - 3 = x$$

$$[g_1(x)]^4 = 3 + x - 2x^2$$

$$x^4 = 3 - x - 2x^2 \quad \Rightarrow \quad f(x) = 0$$

So the solution to the fixed point problem solves the root problem.