MAT360 Section Summary: 2.4
Error Analysis for Iterative Methods

Summary

Alex asked if there's a good way of getting a handle on the number of terms in Newton's method (in problem #5 of 2.3 he discovered that the answers in the back of the text were given to more accuracy than 10^{-4} required). That's the subject of this section.

We learned a bit previously in section 2.2: in 2.2 we obtained useful bounds for fixed-point methods, e.g.

$$|p_n - p| \leq \frac{k^n}{1-k}|p_1 - p_0| \tag{1}$$

where $g(x) \in [a,b] \forall x \in [a,b]$, and $|g'(x)| \leq k < 1$ on $[a,b]$, which brackets the fixed point p. You can use this for Newton's method, but perhaps we can do better, since the convergence is better (I've asserted that it's "quadratic", rather than linear).

Theorem 2.5 (from section 2.3): Let $f \in C^2[a,b]$. If $p \in [a,b]$ is such that $f(p) = 0$ and $f'(p) \neq 0$, then $\exists \delta > 0$ such that Newton's method generates a sequence $\{p_n\}_{n=1}^{\infty}$ converging to p for any initial approximation $p_0 \in [p-\delta, p+\delta]$.

This result is "obvious" (I claimed, in 2.2), since

$$|p_{n+1} - p| \approx \frac{1}{2}|g''(p)||p_n - p|^2 = \left(\frac{1}{2}|g''(p)||p_n - p|\right)|p_n - p|$$

when p_n gets into close proximity (i.e. a δ-neighborhood) of p. We can be assured of "contracting" as long as the magnitude of $g''(x)$ is bounded (e.g. $|g''(x)| < M$) in that neighborhood, so long as

$$\frac{1}{2}M|p_n - p| < 1$$

It's obviously true when $p_n = p$, and we simply choose $|p_n - p| < \frac{2}{M}$ to be assured that we'll converge by the Fixed-Point Theorem (2.3).

Definition 2.6: Suppose that $\{p_n\}_{n=0}^{\infty}$ is a sequence that converges to p, with $p_n \neq p$ for all n. If positive constants λ and α exist with

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

then the sequence **converges to p of order α, with asymptotic error constant λ.**

1. If $\alpha = 1$, the sequence is **linearly convergent** *(e.g. standard convergent fixed point function, with $g'(p) \neq 0$)*, whereas
2. if $\alpha = 2$, the sequence is **quadratically convergent** *(e.g. Newton's method, with $g'(p) \neq 0$)*.

Q: What does *asymptotic* mean?
Q: Is bisection linearly convergent?[1] Contrast this with Exercise #9, for your homework. on $[0,1]$

$p = .1010\overline{10}$ base 2

Theorem 2.7: Let $g \in C[a,b]$ be such that $g(x) \in [a,b] \forall x \in [a,b]$. Suppose, in addition, that g' is continuous on (a,b) and a positive constant $k < 1$ exists with

$$|g'(x)| \leq k$$

$\forall x \in (a,b)$. If $g'(p) \neq 0$, then for any number p_0 in $[a,b]$, the sequence of iterates

$$\boxed{p_n = g(p_{n-1})}$$

[1]The Bisection Algorithm is Not Linearly Convergent. Sui-Sun Cheng and Tzon-Tzer Lu, College Math Journal: Volume 16, Number 1, (1985), Pages: 56-57.

for $n \geq 1$ converges only linearly to the unique fixed point $p \in [a, b]$.

Proof (by the MVT)

$$
\begin{aligned}
p_{n+1} - p &= g(p_n) - g(p) \\
&= g'(\xi_n)\,(p_n - p)
\end{aligned}
$$

$$
\left(\frac{g(p_n) - g(p)}{p_n - p} = g'(\xi_n) \quad \text{where } \xi_n \text{ is between } p_n + p \right)
$$

$$
\lim_{n \to \infty} p_n = p
$$

$$
\boxed{\text{Mike}^{*} \text{ Squeeze Theorem}}
$$

$$
\begin{aligned}
\lim_{n \to \infty} \frac{p_{n+1} - p}{p_n - p} &= \lim_{n \to \infty} g'(\xi_n) \\
&= g'(p)
\end{aligned}
$$

$$
\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^{1}} = |g'(p)| < 1
$$

L $\qquad$ λ

Theorem 2.8: Let p be a solution of the equation $x = g(x)$. Suppose that $g'(p) = 0$ and g'' is continuous and strictly bounded by M on an open interval I containing p. Then $\exists \delta > 0$ such that, for $p_0 \in [p - \delta, p + \delta]$, the sequence $\{p_n = g(p_{n-1})\}_{n=1}^{\infty}$ converges at least quadratically to p. Moreover, for sufficiently large values of n,

$$|p_{n+1} - p| < \frac{M}{2}|p_n - p|^2$$

(Hence, Newton's method is quadratic.)

Proof (by Taylor series, and Fixed-Point theorem)

$$\cancel{p_{n+1} = p + g'(p)(x-p) + \frac{g''(p)}{2}(x-p)^2}$$

$$p_{n+1} = g(p_n) = \underbrace{g(p)}_{p} + \underbrace{g'(p)}_{0}(p_n - p) + \frac{g''(\xi_n)}{2}(p_n - p)^2$$

ξ_n between p_n + p

$$p_{n+1} - p = \frac{g''(\xi_n)}{2}(p_n - p)^2$$

$$\lim_{n \to \infty} \frac{p_{n+1} - p}{(p_n - p)^2} = \lim_{n \to \infty} \frac{g''(\xi_n)}{2} = \frac{g''(p)}{2}$$

by squeeze Thm + continuity of g''.

$$\lim_{n \to \infty} \frac{|p_{n+1} - p|}{|p_n - p|^2} < \frac{M}{2}$$

Example: Here's where we can make use of the quadratic convergence to address Alex's question: For problem #5b, for example, with

$$f(x) = x^3 + 3x^2 - 1$$

$p_0 = 3$ and a solution $p_3 = -2.87939$, we use

$$g(x) = x - \frac{x^3 + 3x^2 - 1}{3x^2 + 6x}$$

and then compute the first and second derivatives of g. We note that by theorem 2.2 there is a unique fixed point in the interval $[-3, -2.74]$; also we see that g has a maximum value of $|g''(x)| \quad 2.5$ on the interval $[-3, -2.74]$. g has a maximum value of $\quad -.27$ on the interval, so we could use Equation (1) above to make our estimate (it gives 8 iterations).

We can do better, of course!

Theorem: the secant method is of order the golden mean.

Motivation: #12

$$|p_{n+1} - p| \approx C\,|p_n - p|\,|p_{n-1} - p|$$

Assume $\{p_n\}$ converges of order α; that is

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} = \lambda$$

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^{\alpha}} \approx \lim_{n\to\infty} C \frac{|p_{n-1} - p|}{|p_n - p|^{\alpha - 1}} = C \lim_{n\to\infty} \frac{1}{\underbrace{\frac{|p_n - p|^{\alpha-1}}{|p_{n-1} - p|}}_{\text{converges to a constant}}}$$

$$1/\alpha = \frac{\alpha - 1}{1} \Rightarrow \alpha^2 - \alpha - 1 = 0$$

$$\frac{1 \pm \sqrt{4+1}}{2} = \frac{1 \pm \sqrt{5}}{2} \qquad \therefore \boxed{\alpha = \frac{1+\sqrt{5}}{2}} \simeq 1.618$$

Definition 2.9: A solution p of $f(x) = 0$ is a **zero of multiplicity** m of f if, for $x \neq p$, we can write $f(x) = (x-p)^m q(x)$, where $\lim_{x \to p} q(x) \neq 0$.

Theorem 2.10: $f \in C^1[a,b]$ has a simple zero at $p \in (a,b) \iff f(p) = 0$, but $f'(p) \neq 0$.

Theorem 2.11: $f \in C^m[a,b]$ has a zero of multiplicity m at $p \in (a,b) \iff 0 = f(p) = f'(p) = \ldots = f^{(m-1)}(p)$, but $f^{(m)}(p) \neq 0$.

#11 $\quad g(x) = x - \varphi(x) f(x)$ where $f(p) = 0$

Want $g'(p) = 0$:

$$g'(x)\Big|_{x=p} = 1 - \underbrace{\varphi'(x) f(x)} - \varphi(x) f'(x)\Big|_{x=p} = 0$$

$$1 - 0 - \varphi(p) f'(p) = 0$$

If we can choose φ / $\varphi(p) = \frac{1}{f'(p)}$

Then $g'(p) = 0$. Choose $\varphi(x) = \frac{1}{f'(x)}$

(Newton's choice).

But, as B&K* notes, only works when $f'(p) \neq 0$.

Demand also that $g''(p) = 0$:

$$-\cancel{\varphi''(x) f(x)}^{\,0} - \varphi'(x) f'(x) - \varphi'(x) f'(x) - \varphi(x) f''(x)\Big|_{x=p} = 0$$

$$\varphi(p) = \frac{-2f'(p)\,\varphi'(p)}{f''(p)} = \underbrace{\frac{1}{f'(p)}}_{\text{from } g'(p)=0}$$

$\therefore \quad \varphi'(p) = \dfrac{f''(p)}{-2f'(p)^2}$

So $\quad \varphi(p) = \dfrac{1}{f'(p)} \quad \& \quad \varphi'(p) = \dfrac{f''(p)}{-2f'(p)^2}$

Assume $\quad \varphi(x) = \dfrac{1}{f'(x)} + \psi(x) f(x)$

$$\varphi'(x) = -\frac{f''(x)}{(f'(x))^2} + \psi'(x) f(x) + \psi(x) f'(x)$$

$$\varphi'(p) = -\frac{f''(p)}{(f'(p))^2} + \cancel{\psi'(p) f(p)} + \psi(p) f'(p)$$

demand $= \dfrac{f''(p)}{-2f'(p)^2}$, & solve for $\psi(p)$:

$$\psi(p) = \frac{1}{2} \frac{f''(p)}{(f'(p))^2} \cdot \frac{1}{f'(p)}$$

One choice:

$$\psi(x) = \frac{f''(x)}{2(f'(x))^3}, \implies$$

$$\varphi(x) = \frac{1}{f'(x)} + \left(\frac{f''(x)}{2(f'(x))^3}\right) f(x), \implies$$

$$g(x) = x - \left[\frac{1}{f'(x)} + \frac{f''(x) f(x)}{2(f'(x))^3}\right] f(x), \text{ or}$$

$$\boxed{g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)}{2f'(x)} \left[\frac{f(x)}{f'(x)}\right]^2}$$

Compare #11